Module 10: Causal Mechanisms

Fall 2021

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Gov 2003 (Harvard)

1/ Causal Mechanisms

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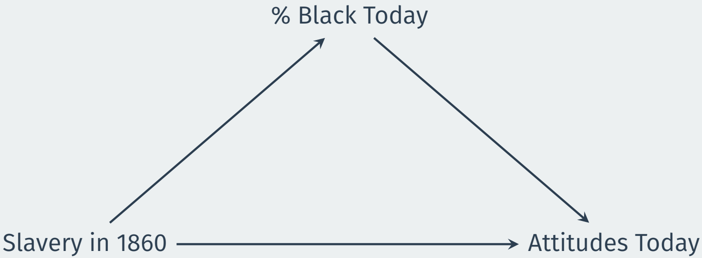
- How to adjudicate between theories that predict the same ATE?
- Put differently: what **mechanism** drives a particular causal effect?

Example: Deep Roots



- Effect of antebellum slavery on modern white attitudes:

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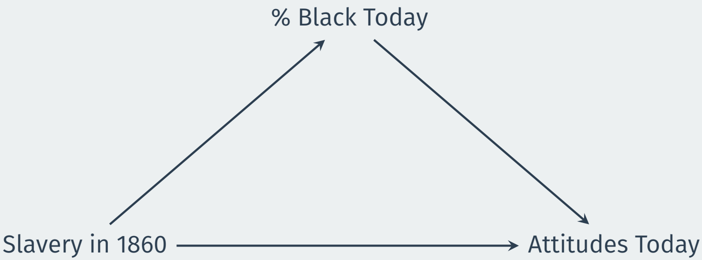

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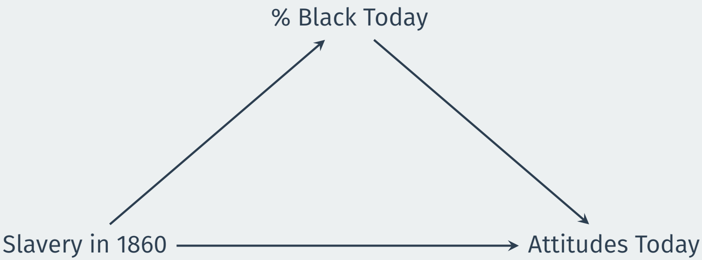

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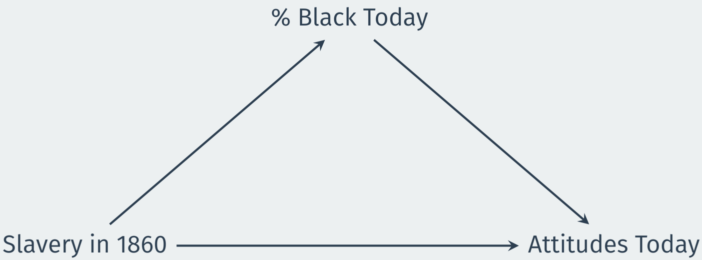

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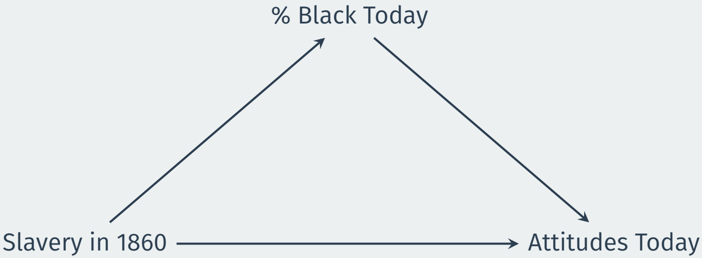

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    - Whites living in formerly enslaved areas in the South today more likely to be conservative on racial issues.
- Two possible mechanisms with very different implications:
    - Historical persistence of attitudes via intergenerational transfer.
    - Or is this effect due to demographic persistence? (More African Americans in former enslaved areas today $\rightsquigarrow$ whites threatened today)

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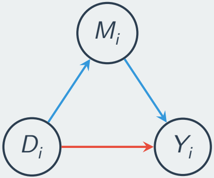
- DAG representation:



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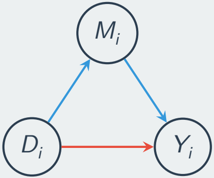

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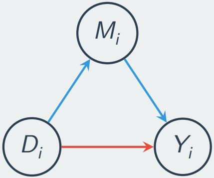

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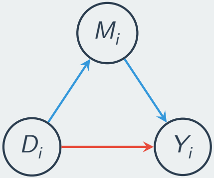

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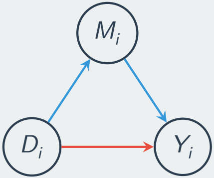

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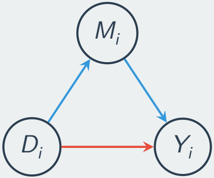

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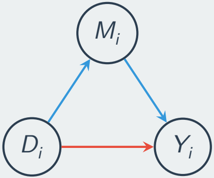

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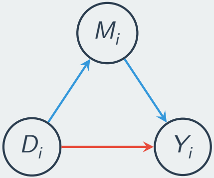

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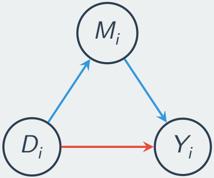

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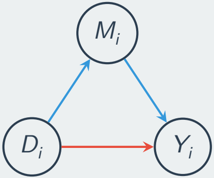

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- Consistency: $M_i = M_i(D_i)$ and $Y_i(D_i, M_i(D_i))$.

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  - Not just the fundamental problem of CI.

2/ Estimands

Controlled direct effects (CDE)

- Definition for each $m \in \mathcal{M}$:

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- If M_i fully mediates effect of D, then CDEs will be 0 for all m.
  - $\rightsquigarrow$ can be used to establish existence of unmediated path from $D \to Y$.
- Can capture **interactions** if $\overline{\zeta}_i(m) \neq \overline{\zeta}_i(m')$

Natural indirect effects (NIE)

- Definition of the **natural indirect effect** (NIE):

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    - Holding fixed the value of treatment.
- Also called the **causal mediation effect**
- If D_i doesn't affect M_i, so that $M_i(1) = M_i(0)$, then $\delta_i = 0$.

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- Definition of the **natural direct effect** (NDE) of the treatment:

$$
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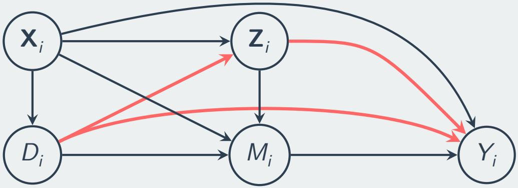
- Total effect decomposition:

$$\tau_i = Y_i(1, M_i(1)) - Y_i(0, M_i(0)) = \underbrace{\delta_i(d)}_{\text{NIE}} + \underbrace{\zeta_i(1-d)}_{\text{NDE}}$$

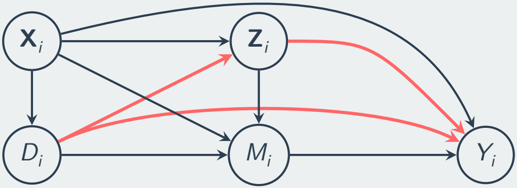
3/ Identification

Identification for CDEs



- Conditioning sets:

Identification for CDEs



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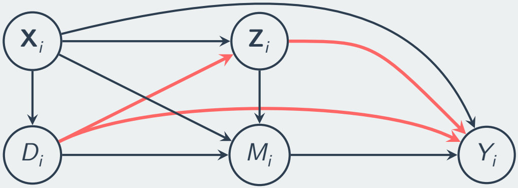

- Conditioning sets:
  - $\mathbf{X}_i$: pre-treatment confounders
  - $\mathbf{Z}_i$: post-treatment or intermediate confounders

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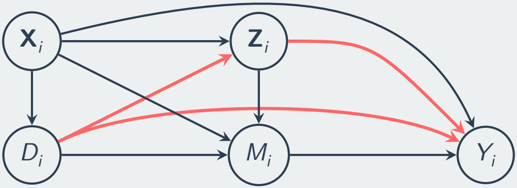

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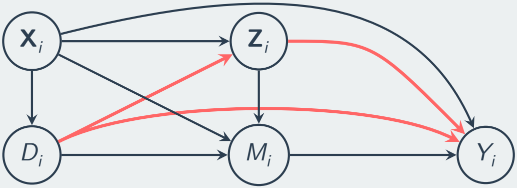

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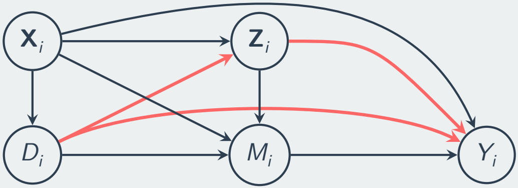

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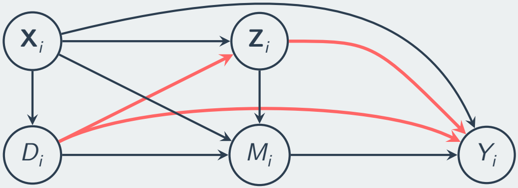

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Identifying the ACDE

- Post-treatment bias if we just condition on $\mathbf{Z}_i$:

$$\overline{\xi}(m) \neq \sum_{\mathbf{x},\mathbf{z}} \{\mathbb{E}[Y_i \mid D_i = 1, M_i = m, \mathbf{X}_i = \mathbf{x}, \mathbf{Z}_i = \mathbf{z}] - \mathbb{E}[Y_i \mid D_i = 0, M_i = m, \mathbf{X}_i = \mathbf{x}, \mathbf{Z}_i = \mathbf{z}]\}\mathbb{P}(\mathbf{X}_i = \mathbf{x}, \mathbf{Z}_i = \mathbf{z})$$

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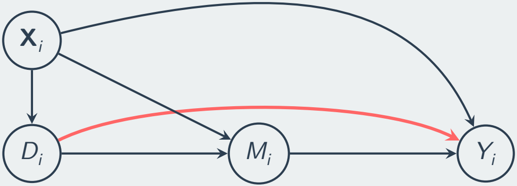
- **g-formula** (Robins) generalizes to any number of treatments

Identification for mediation



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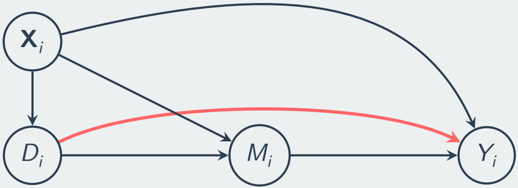

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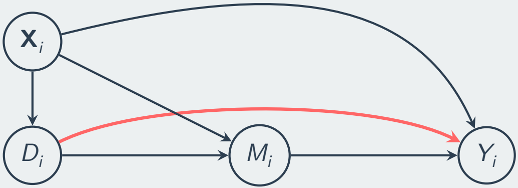

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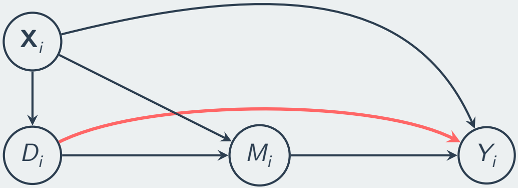

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  - Assumes away post-treatment bias conditioning on M_i

Identifying (in)direct effects

- ANIE under binary treatment/mediator:

$$\bar{\delta}(d) = \sum_{\mathbf{x}} \Big(\underbrace{\{\mathbb{P}[M_i = 1 \mid D_i = 1, \mathbf{X}_i = \mathbf{x}] - \mathbb{P}[M_i = 1 \mid D_i = 0, \mathbf{X}_i = \mathbf{x}]\}}_{\text{effect of } D_i \text{ on } M_i}$$

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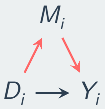
- Multiply paths given $\mathbf{X}_i$ and aggregate intuitive given DAG:



(In)direct effects with non-binary mediators

- Let's say that the mediator has J categories:

$$\overline{\delta}(d) = \sum_{\mathbf{x}} \Bigg(\sum_{m=0}^{J-1} \mathbb{E}[Y_i \mid M_i = m, D_i = d, \mathbf{X}_i = \mathbf{x}] \\ \times \{\mathbb{P}[M_i = m \mid D_i = 1, \mathbf{X}_i = \mathbf{x}] - \mathbb{P}[M_i = m \mid D_i = 0, \mathbf{X}_i = \mathbf{x}]\} \Bigg) \\ \times \mathbb{P}(\mathbf{X}_i = \mathbf{x})$$

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  - Effect of D_i for a fixed m averaged over the distribution of M_i when $D_i = d$.

Alternative identification

- Robins proposed a different identification strategy, based on a **no-interactions assumption**:

$$Y_i(1, m) - Y_i(0, m) = Y_i(1, m') - Y_i(0, m')$$

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- $\rightsquigarrow$ ACDE = ANDE.
- Strong assumption because it has to hold at the individual level (like monotonicity for IV).

4/ Linear Structural Equation Models

Estimation

- Let's say that we have a linear, structural model for all variables:

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Linear models and mediation

- It's clear that we can write the total effect of the treatment in the following way:

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- What about the indirect effect:

$$\begin{aligned} Y_i(0, M_i(1)) - Y_i(0, M_i(0)) =& \beta_0 + \beta_2(\alpha_0 + \alpha_1 + \eta_i) + \varepsilon_i \\ & - \beta_0 - \beta_2(\alpha_0 + \eta_i) - \varepsilon_i \\ =& \beta_2 \cdot \alpha_1 \end{aligned}$$

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- Indirect effect as the product: $\widehat{ANIE} = \hat{\alpha}_1 \hat{\beta}_2$.

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- **Implicit assumption**: no interactions

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- We could incorporate an interaction into the model here to allow for the indirect effect to vary.

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- We can use this formula to estimate standard errors for the indirect effects.

5/ Nonparametric Estimation

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- Same for M_i:

$$\widehat{\mathbb{P}}[M_i = m \mid D_i = d, \mathbf{X}_i = \mathbf{x}] = \frac{\sum_i \mathbb{1}\{M_i = m, D_i = d, \mathbf{X}_i = \mathbf{x}\}}{\sum_i \mathbb{1}\{D_i = d, \mathbf{X}_i = \mathbf{x}\}}$$

What about more complicated scenarios?

- If the number of categories is large, then we can use nonparametric regressions for the outcome and the mediator.

$$\mu_{dm}(\mathbf{x}) = \mathbb{E}[Y_i \mid M_i = m, D_i = d, \mathbf{X}_i = \mathbf{x}]$$

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- Flexibly estimate $\mu_{dm}(\mathbf{x})$ as a function of $\mathbf{x}$ using splines of $\mathbf{x}$.

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- Need to be careful with the curse of dimensionality in $\mathbf{X}_i$. Use good nonparametric strategies (cross-validation, etc)

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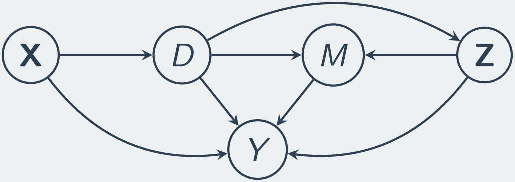
- Modeling M_i probably appropriate here.

6/ Estimating Controlled Direct Effects

Sequential g-estimation



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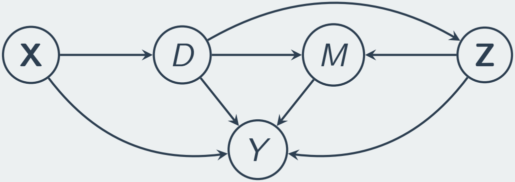

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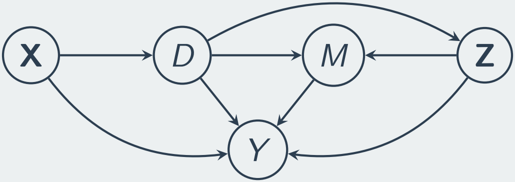

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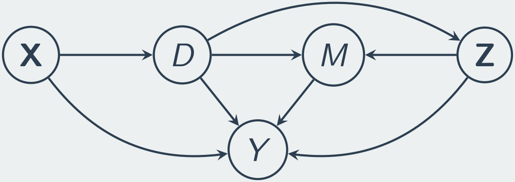

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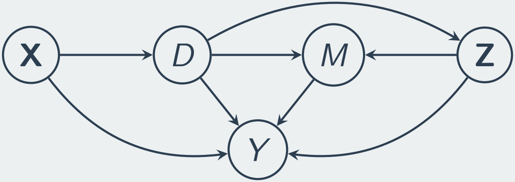

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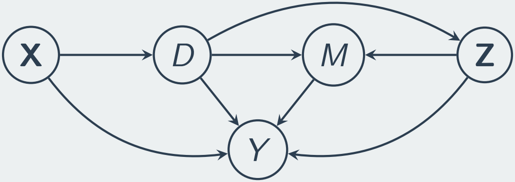

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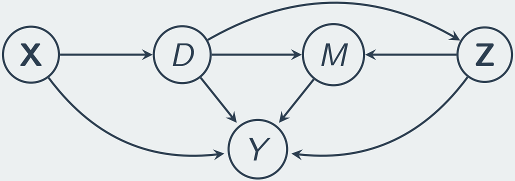
- γ_2 **is** the effect of M_i on Y_i (if model is correct)

Blip down



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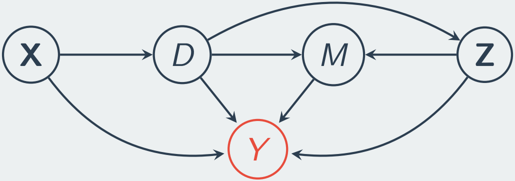

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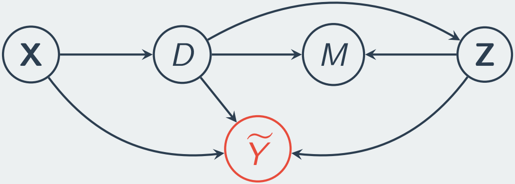

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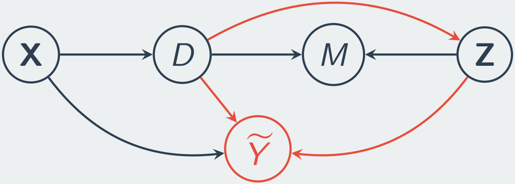

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- ATE - ACDE $\neq$ an indirect effect, but still can tell us something about mechanisms.

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