Module 9: Panel Data

Fall 2021

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Gov 2003 (Harvard)

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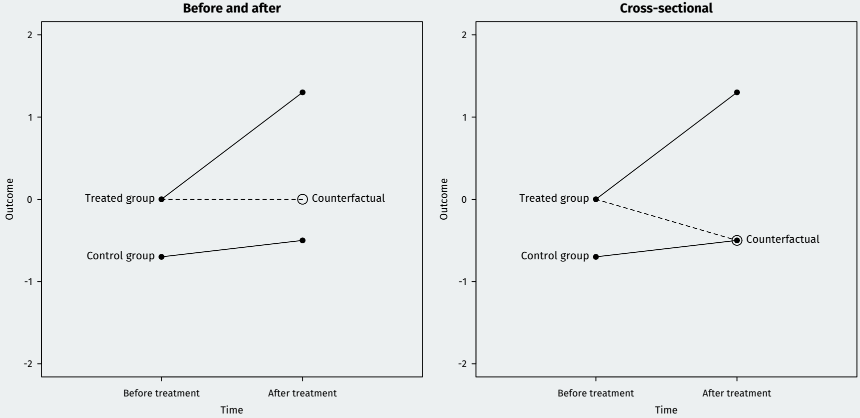
- Now there are two possible sources of variation to exploit:
    - Exploit **cross-sectional** variation in treatment.
    - Exploit variation in treatment **within a unit over time** (before/after)

Cross-sectional vs before/after



1/ Difference in differences

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- Part (a) is just a conditional average of observed data $\rightsquigarrow$ identified.
- Part (b) is a counterfactual: what would the average outcome in the treated group have been if it have been in control?

Three control strategies

$$\tau_{ATT} = \mathbb{E}[Y_{i1}(1)|G_i = 1] - \mathbb{E}[Y_{i1}(0)|G_i = 1]$$

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- **Cross-sectional design**
    - Assumption: mean independence of treatment

$$\mathbb{E}[Y_{i1}(0)|G_i = 1] = \mathbb{E}[Y_{i1}(0)|G_i = 0]$$

    - Use post-treatment control group:

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- **Before-and-after design**
  - Assumption: no trends

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  - Use pre-period outcome in treated group:

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- **Difference-in-differences**:
    - Assumption: parallel trends

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    - Use pre-period treated outcome plus trend in control group:

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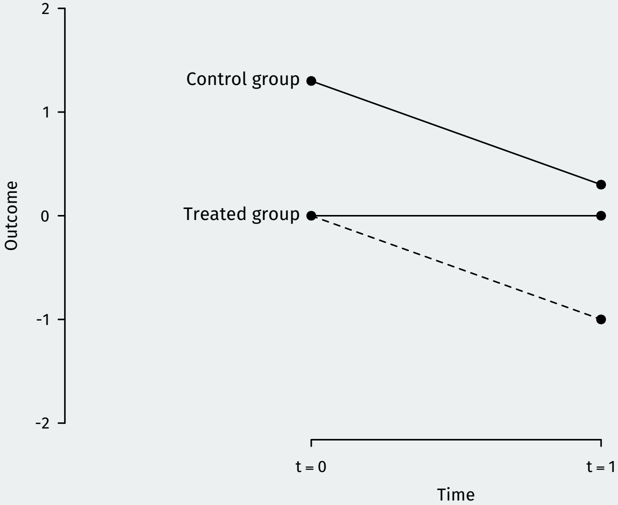
- Not invariant to nonlinear transformations!
    - Parallel trends for Y_{it} implies non-parallel trends for $\log(Y_{it})$ and vice versa.

Parallel trends in a graph



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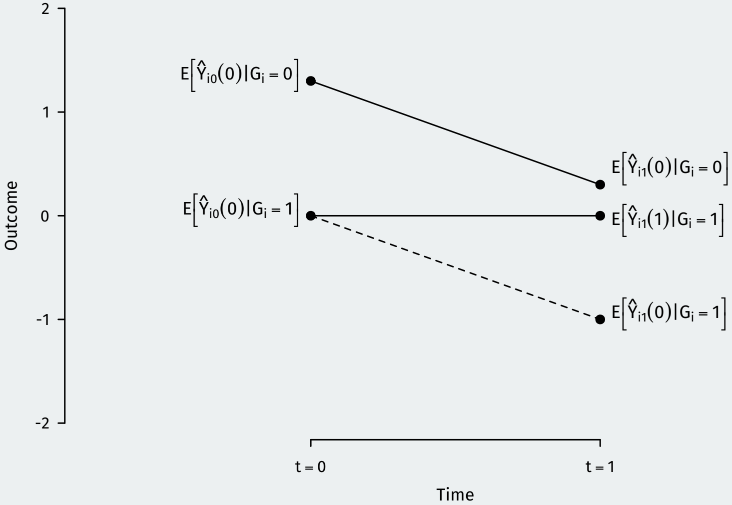

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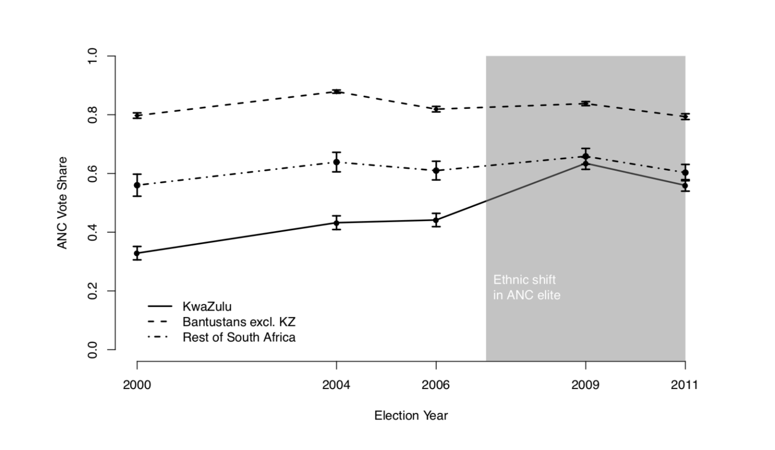
    - **Ashenfelter's dip**: empirical finding that people who enroll in job training programs see their earnings decline prior to that training.
- Falsification test: check pre-treatment parallel trends.
    - Doesn't imply parallel trends hold for the post-period however!

Checking parallel trends (de Kadt/Larreguy, 2018)



Estimation

- Estimation with panel data:

$$\widehat{\tau}_{\text{ATT}} = \underbrace{\frac{1}{n_1}\sum_{i=1}^{n} G_i \left\{Y_{i1} - Y_{i0}\right\}}_{\text{average trend in treated group}} - \underbrace{\frac{1}{n_0}\sum_{i=1}^{n}(1 - G_i) \left\{Y_{i1} - Y_{i0}\right\}}_{\text{average trend in the control group}}$$

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    - Regress $\Delta Y_i = Y_{i1} - Y_{i0}$ on G_i.

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- Also possible to use DID on repeated cross sections.

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  - Basically, TWFE is an odd weighted average of DID effects with sometimes negative weights.

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  - LDV: previous outcome directly affects treatment assignment.

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- Estimator: estimate CEF $\mathbb{E}[Y_{i1} \mid Y_{i0}, G_i] = \alpha + \rho Y_{i0} + \tau G_i$

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- **Matching**: conduct DID analysis on units with similar values of $\mathbf{X}_i$

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    - Possible model misspecification!

2/ Fixed effects

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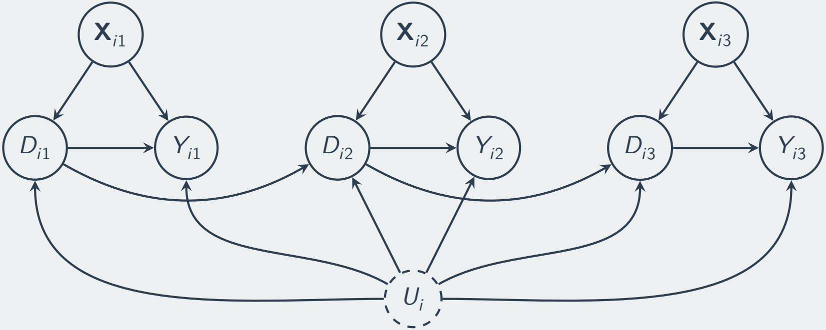
  - $\rightsquigarrow$ lags of treatments become part of time-varying confounders.

Strict exogeneity DAG



Strict exogeneity implied by strict ignorability $Y_{it}(d) \perp\!\!\!\perp \overline{D}_i \mid \overline{X}_i, U_i$

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  - Latter allows for substantial serial dependence in the original errors.

Estimation notes

- Within estimator can be implemented by adding unit dummy variables.

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    - `{fixest}` in R, `-reghdfe-` in Stata

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    - bias correction: estimate the bias and subtract it off (valid for nonlinear models too).

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    - Bias from incidental parameters, but disappears as $T \to \infty$

3/ Synthetic control methods

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    - Choose the weights that minimize the pretreatment differences between treated and synthetic control.

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	Time period						
	1	2	⋯	T_0	$T_0 + 1$	⋯	T
Treated unit ($i = 1$)	0	0	0	0	1	1	1
Control group ($i = 2, \ldots, J + 1$)	0	0	0	0	0	0	0

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- Goal: estimate $\left(\tau_{1,T_0+1}, \dots, \tau_{1,T}\right)$.

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  - Can also add a penalty for how dispersed the weights are.
- We hope this implies for $t > T_0$: $\sum_{j=2}^{J+1} w_j Y_{jt} \approx Y_{1t}(0)$

Without synthetic controls

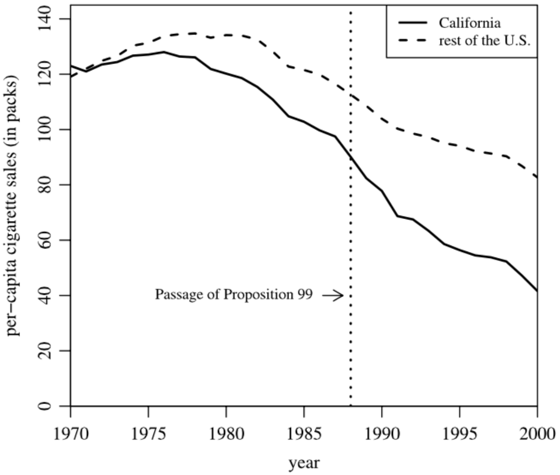

Figure 1. Trends in per-capita cigarette sales: California vs. the rest of the United States.

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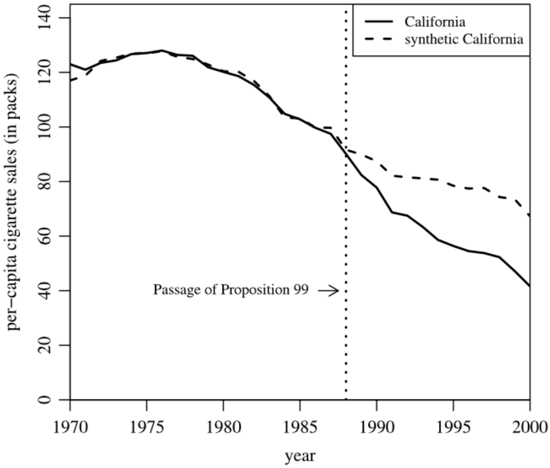

Figure 2. Trends in per-capita cigarette sales: California vs. synthetic California.

Weights

Table 2. State weights in the synthetic California

State	Weight	State	Weight
Alabama	0	Montana	0.199
Alaska	–	Nebraska	0
Arizona	–	Nevada	0.234
Arkansas	0	New Hampshire	0
Colorado	0.164	New Jersey	–
Connecticut	0.069	New Mexico	0
Delaware	0	New York	–
District of Columbia	–	North Carolina	0
Florida	–	North Dakota	0
Georgia	0	Ohio	0
Hawaii	–	Oklahoma	0
Idaho	0	Oregon	–
Illinois	0	Pennsylvania	0
Indiana	0	Rhode Island	0
Iowa	0	South Carolina	0
Kansas	0	South Dakota	0
Kentucky	0	Tennessee	0
Louisiana	0	Texas	0
Maine	0	Utah	0.334
Maryland	–	Vermont	0
Massachusetts	–	Virginia	0
Michigan	–	Washington	–
Minnesota	0	West Virginia	0
Mississippi	0	Wisconsin	0
Missouri	0	Wyoming	0

Inference

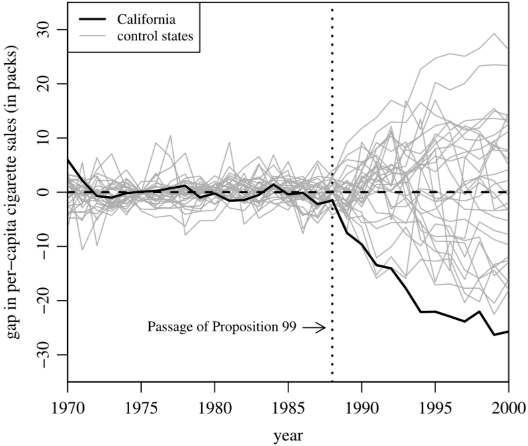

Figure 6. Per-capita cigarette sales gaps in California and placebo gaps in 29 control states (discards states with pre-Proposition 99 MSPE five times higher than California's).

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- Either fixed effects OR lagged dependent variables, not both.

SCM properties

- Suppose perfect balancing weights exist $(w_2^*, \dots, w_{J+1}^*)$ such that:

$$\sum_{j=2}^{J+1} w_j^* Y_{jt} = Y_{1t} \qquad \sum_{j=2}^{J+1} w_j^* \mathbf{X}_j = \mathbf{X}_i$$

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- Outside of those models: ?????

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