Module 6: Noncompliance and Instrumental Variables

Fall 2021

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Gov 2003 (Harvard)

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    - First: motivate IV through experiments and noncompliance.
    - Then: how does this relate to classical econometric methods like TSLS?

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  - Full compliance means $Z_i = D_i$ for all i

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- Alternative: leverage latent strata of **compliance types**

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  - But no one receives treatment if in control ($D_i(0) = 0, \ \forall i$)
  - **Two-sided noncompliance** is when you can refuse to comply with treatment **or** control.

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- Consistency assumption: $Y_i = Y_i(Z_i, D_i(Z_i))$

Some notation

- Let's use 0/1 subscripts for assignment and t/c subscripts for uptake:

$$n_1 = \sum_{i=1}^{n} Z_i \qquad n_0 = \sum_{i=1}^{n} 1 - Z_i \qquad n_t = \sum_{i=1}^{n} D_i \qquad n_c = \sum_{i=1}^{n} 1 - D_i$$

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    - For observational uses of IV, might condition on some $\mathbf{X}_i$.

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    - Effect of D_i is maybe more externally valid than Z_i.

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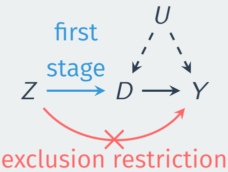
- Implies overall ITT on uptake is equal to the **proportion of compliers**

$$\text{ITT}_D = \pi_{\text{co}}\text{ITT}_{D,\text{co}} + \pi_{\text{nc}}\text{ITT}_{D,\text{nc}} = \pi_{\text{co}}$$

3/ Instrumental variables

Exclusion restriction



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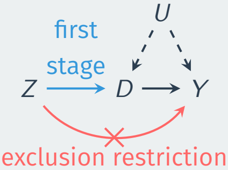

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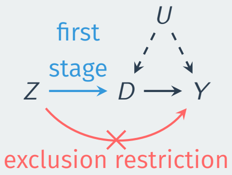

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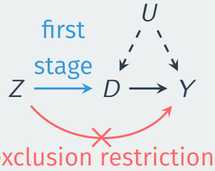

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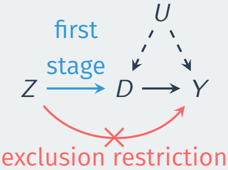

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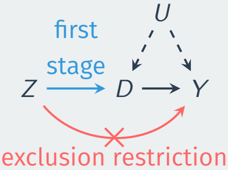

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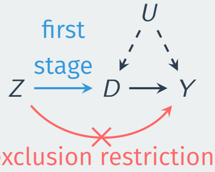

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  - $Y_i(z, d) = Y_i(z', d)$ for all z, z' and d.
  - Assignment to canvassing only affects turnout through actual canvassing.
  - Not a testable assumption and can't be guaranteed by design.
- Implies that potential outcomes only a function of D_i:

$$Y_i(1) = Y_i(D_i = 1) = Y_i(Z_i = 1, D_i = 1) = Y_i(Z_i = 1, D_i = 1)$$
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- Allows us to connect the ITT on the outcome to compliance groups:

$$\text{ITT}_Y = \pi_{\text{co}}\text{ITT}_{Y,\text{co}} + \pi_{\text{nc}}\text{ITT}_{Y,\text{nc}} = \text{ITT}_D\text{ITT}_{Y,\text{co}}$$

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- **LATE Theorem** under one-sided noncompliance, exclusion restriction, first-stage, and randomization:

$$\tau_{\text{LATE}} = \text{ITT}_{Y,\text{co}} = \frac{\text{ITT}_Y}{\text{ITT}_D}$$

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- We can use the delta method to find the (superpopulation) variance:

$$\mathbb{V}[\widehat{\tau}_{iv}] = \frac{1}{\text{ITT}_D^2}\mathbb{V}\left[\widehat{\text{ITT}}_Y\right] + \frac{\text{ITT}_Y^2}{\text{ITT}_D^4}\mathbb{V}\left[\widehat{\text{ITT}}_D\right] - 2\frac{\text{ITT}_Y}{\text{ITT}_D^3}\text{cov}\left[\widehat{\text{ITT}}_Y, \widehat{\text{ITT}}_D\right]$$

4/ Two-sided noncompliance

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- Z_i is the encouragement and D_i is the treatment.
- No change in estimation, just different identification assumptions.

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$$\begin{aligned}\text{ITT}_Y &= \text{ITT}_{Y,\text{co}}\pi_{\text{co}} + \underbrace{\text{ITT}_{Y,\text{at}}}_{=0 \text{ (ER)}}\pi_{\text{at}} + \underbrace{\text{ITT}_{Y,\text{nt}}}_{=0 \text{ (ER)}}\pi_{\text{nt}} + \text{ITT}_{Y,\text{de}}\underbrace{\pi_{\text{de}}}_{=0 \text{ (mono)}} \\ &= \text{ITT}_{\text{co}}\pi_{\text{co}}\end{aligned}$$

- $\rightsquigarrow$ same identification result: $\tau_{\text{LATE}} = \text{ITT}_Y / \text{ITT}_D$

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    - Alternative: bound the ATE?

5/ Basic two-stage least squares

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- This implies the following CEF form for Y_i conditional on Z_i:

$$\mathbb{E}[Y_i \mid Z_i] = \alpha + \tau \mathbb{E}[D_i \mid Z_i] = \alpha + \tau \cdot (\gamma Z_i)$$

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    - Under this model, $\widehat{\tau}_{2SLS} \overset{p}{\rightarrow} \tau$ (but don't use SEs from second stage)

Binary treatment and instrument

- Under binary treatment/instrument, TSLS estimand is the LATE:

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  - Otherwise, τ is an odd weighted function of causal effects and $\tau \neq \tau_{\text{LATE}}$

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    - AR test statistic: $g(\tau_0)^2 / \Omega(\tau_0) \sim \chi^2$ no matter first-stage effect.

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$$g(\tau_0) = \widehat{\text{ITT}}_Y - \widehat{\text{ITT}}_D \tau_0 \sim N(0, \Omega(\tau_0))$$
$$\Omega(\tau_0) = \mathbb{V}[\widehat{\text{ITT}}_Y] + \tau_0^2 \mathbb{V}[\widehat{\text{ITT}}_D] - 2\tau_0 \text{cov}(\widehat{\text{ITT}}_Y, \widehat{\text{ITT}}_D)$$

  - AR test statistic: $g(\tau_0)^2 / \Omega(\tau_0) \sim \chi^2$ no matter first-stage effect.
  - Can invert (analytically!) to get confidence intervals

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    - 5 unknowns and 4 knowns under monotonicity.

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    - If instrument can only increase by 1 dose, then simplifies to weighted average of principal strata effects.

6/ General two-stage least squares

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Y_i &= \mathbf{X}_i'\beta + \varepsilon_i \\
\mathbf{\Pi}'\mathbf{Z}_i Y_i &= \mathbf{\Pi}'\mathbf{Z}_i\mathbf{X}_i'\beta + \mathbf{\Pi}'\mathbf{Z}_i\varepsilon_i \\
\mathbb{E}[\mathbf{\Pi}'\mathbf{Z}_i Y_i] &= \mathbb{E}[\mathbf{\Pi}'\mathbf{Z}_i\mathbf{X}_i']\beta + \mathbb{E}[\mathbf{\Pi}'\mathbf{Z}_i\varepsilon_i] \\
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\mathbb{E}[\mathbf{\Pi}'\mathbf{Z}_i Y_i] &= \mathbb{E}[\mathbf{\Pi}'\mathbf{Z}_i\mathbf{X}_i']\beta \\
\mathbb{E}[\tilde{\mathbf{X}}_i Y_i] &= \mathbb{E}[\tilde{\mathbf{X}}_i\mathbf{X}_i']\beta \\
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- This is how R/Stata estimates the 2SLS parameters

Asymptotic variance for 2SLS

- We can write the centered, normalized TSLS estimator as:

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- HC2, clutering, and autocorrelation versions exist