Module 4: Linear Regression and Randomized Experiments

Fall 2021

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Gov 2003 (Harvard)

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- Why might we use regression?
    - **Simplicity**: known tool that is already very common.
    - **Increased precision**: we may want to add covariates for more precise effect estimates.

1/ Regression with no covariates

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- Generalizes to discrete treatments with > 2 levels.

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  - Slope $\tau = \mathbb{E}[Y_i(1) - Y_i(0)]$ the PATE.
  - Error is deviation for control PO + treatment effect heterogeneity.

Mean independent errors

$$\varepsilon_i = (Y_i(0) - \mathbb{E}[Y_i(0)]) + D_i \cdot (\tau_i - \tau)$$

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- Randomization + consistency $\rightsquigarrow$ linear model.
    - Does not imply homoskedasticity or normal errors, though!

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- Under homoskedasticity, variance of the OLS estimator is:

$$\mathbb{V}[\widehat{\tau}_{\text{ols}} \mid \mathbf{D}] = \frac{\sigma^2}{\sum_{i=1}^n \left(D_i - \overline{D}\right)^2}$$

Variance estimation

- "Standard" variance estimator under homoskedasticity:

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    - Design is balanced: $n_1 = n_0$

Robust SEs

- Eicker-Huber-White (EHW) robust/sandwich variance estimator:

$$
\begin{aligned}
\widehat{\mathbb{V}}_{\text{EHW}} &= \left(\sum_{i=1}^{n} \mathbf{X}_i \mathbf{X}_i'\right)^{-1} \left(\sum_{i=1}^{n} \hat{\varepsilon}_i^2 \mathbf{X}_i \mathbf{X}_i'\right) \left(\sum_{i=1}^{n} \mathbf{X}_i \mathbf{X}_i'\right)^{-1} \\
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- Recall the PATE-targeted variance of the difference-in-means:

$$\mathbb{V}(\widehat{\tau}_{\text{diff}}) = \frac{\sigma_0^2}{n_0} + \frac{\sigma_1^2}{n_1} = \frac{\mathbb{V}[Y_i(0)]}{n_0} + \frac{\mathbb{V}[Y_i(1)]}{n_1}$$

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- Eicker-Huber-White (EHW) robust/sandwich variance estimator:

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- $\rightsquigarrow$ simple OLS + HC2 = unbiased point and variance estimator.

2/ Linear regression with covariates

Adding covariates

- What if we add covariates to our regression model?

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- Estimation: EHW robust variance estimators are consistent or asymptotically conservative for $\mathbb{V}[\widehat{\tau}_{\text{adj}}]$

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    - Block "fixed effects" $W_{ij} = 1$ if i is in block j, 0 otherwise.
    - Blocks $j \in \{1, \ldots, J\}$ with sizes $w_j = n_j / n$ and propensity scores $p_j = n_{1,j} / n_j$
- Can we just include the block FEs in OLS?

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- Latter two allow for additional covariates to be added.

3/ Cluster randomized experiments

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- Quantity of interest still at individual level:

$$\text{SATE} = \frac{1}{n} \sum_{k=1}^{K} \sum_{i=1}^{m_k} \{Y_{ik}(1) - Y_{ik}(0)\}$$

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- Neyman-style conservative variance:

$$\mathbb{V}[\widehat{\tau}_{\text{cl}} \mid \mathbf{O}] \leq \frac{\mathbb{V}[\overline{Y}_k(1)]}{J_1} + \frac{\mathbb{V}[\overline{Y}_k(0)]}{J_0} \quad \text{where for } d = 0, 1 \quad \overline{Y}_k(d) = \frac{1}{m}\sum_{m=1}^{m} Y_{ik}(d)$$

Cost of clustering

- Standard variance under **individual assignment**:

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