Module 3: Inference for the Average Treatment Effect

Fall 2021

Matthew Blackwell

Gov 2003 (Harvard)

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    - No assumptions to fill in the potential outcomes.
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    - $\rightsquigarrow$ asymptotic approximations.
- What's common: the focus on **randomization** as generating variation in estimators.

Social pressure effect

- Gerber, Green, and Larimer (APSR, 2008)

Dear Registered Voter:

WHAT IF YOUR NEIGHBORS KNEW WHETHER YOU VOTED?

Why do so many people fail to vote? We've been talking about the problem for years, but it only seems to get worse. This year, we're taking a new approach. We're sending this mailing to you and your neighbors to publicize who does and does not vote.

The chart shows the names of some of your neighbors, showing which have voted in the past. After the August 8 election, we intend to mail an updated chart. You and your neighbors will all know who voted and who did not.

DO YOUR CIVIC DUTY—VOTE!

MAPLE DR	Aug 04	Nov 04	Aug 06
9995 JOSEPH JAMES SMITH	Voted	Voted	______
9995 JENNIFER KAY SMITH		Voted	______
9997 RICHARD B JACKSON		Voted	______
9999 KATHY MARIE JACKSON		Voted	______

Social pressure results

TABLE 2. Effects of Four Mail Treatments on Voter Turnout in the August 2006 Primary Election

	Experimental Group				
	Control	Civic Duty	Hawthorne	Self	Neighbors
Percentage Voting	29.7%	31.5%	32.2%	34.5%	37.8%
N of Individuals	191,243	38,218	38,204	38,218	38,201

- Typical reporting of the Neighbors vs Control effect:

$$\text{estimate} = \frac{1}{n_1}\sum_{i=1}^{n} D_i Y_i - \frac{1}{n_0}\sum_{i=1}^{n}(1 - D_i) Y_i \approx 8.1$$

$$\text{standard error} = \sqrt{\frac{\widehat{\sigma}_1^2}{n_1} + \frac{\widehat{\sigma}_0^2}{n_0}} \approx 0.27$$

$$95\% \text{ CI} = [\text{est} - 1.96 \cdot SE,\ \text{est} + 1.96 \cdot SE] \approx [7.57, 8.63]$$

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- Can this analysis be justified by randomization?

1/ Completely randomized experiments

Estimand of interest

- Common estimand in experiments: **sample average treatment effect**

$$\text{SATE} = \tau_{\text{fs}} = \frac{1}{n}\sum_{i=1}^{n}[Y_i(1) - Y_i(0)]$$

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  - the randomization distribution + sampling from the population.

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- Conditional on the sample, $\widehat{\tau}_{\text{diff}}$ only varies because of D_i

Repeated samples/randomizations

randomization 1 C T C T T C

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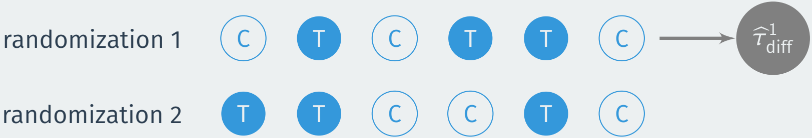

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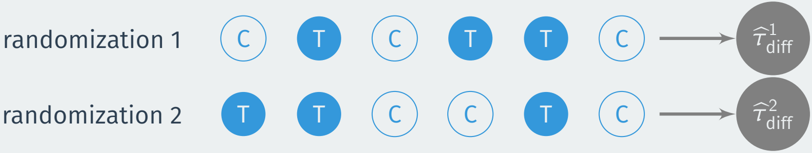

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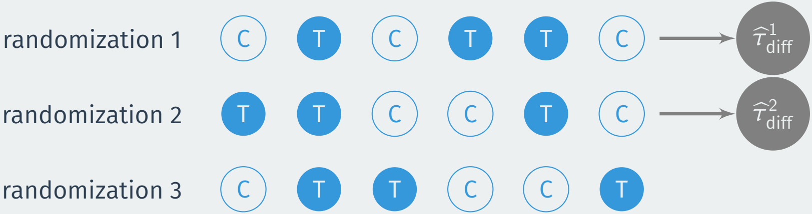

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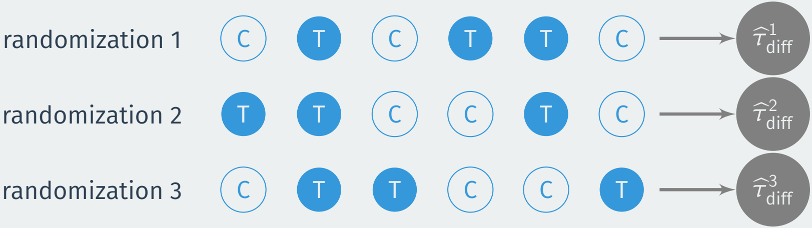

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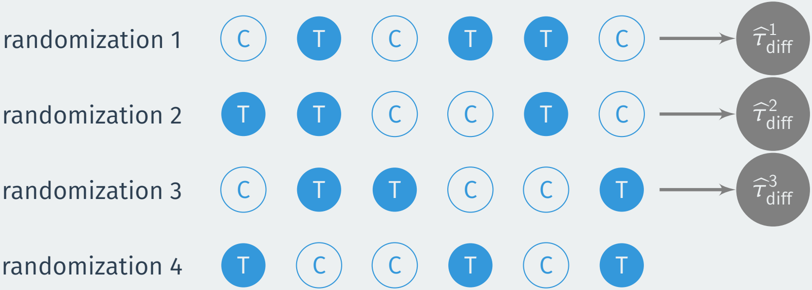

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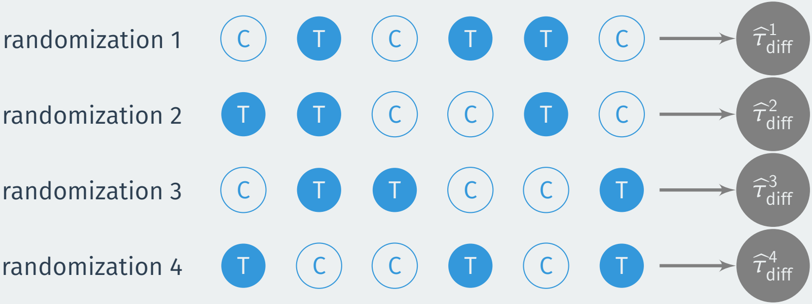

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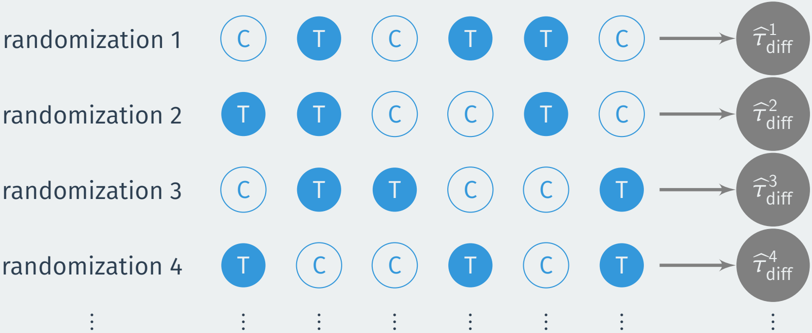

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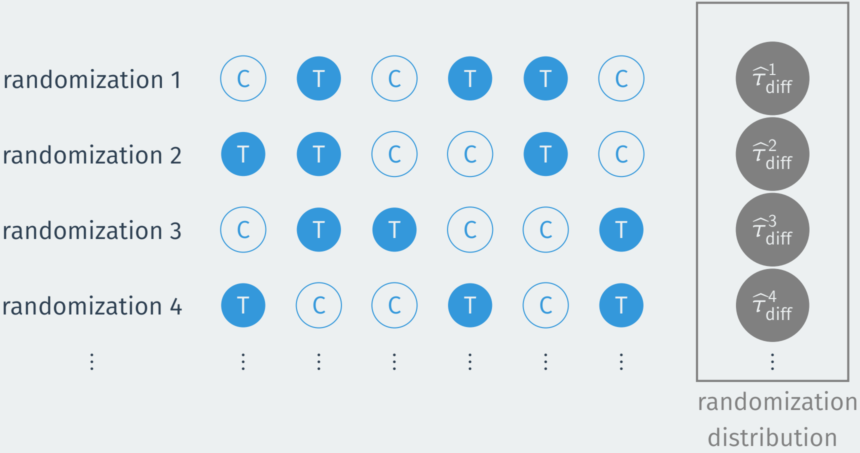

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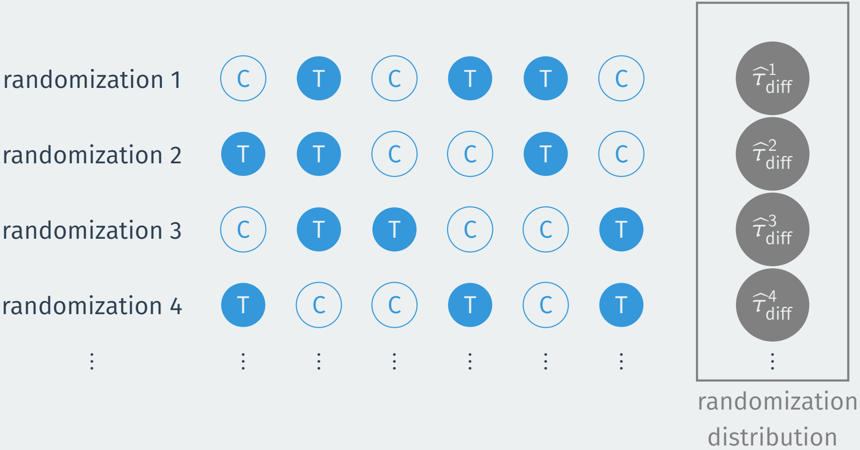

- **Randomization distribution** = sampling distribution of this estimator.

Finite-sample properties

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    - **Unbiasedness**: is mean of the randomization distribution equal to the true SATE?
    - **Sampling variance**: variance of the randomization distribution?
- Use these properties to construct confidence intervals, conduct tests.

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- Note: number treated/control doesn't matter for unbiasedness!

Finite-sample sampling variance

- Sampling variance of the difference-in-means estimator is:

$$\mathbb{V}_D(\widehat{\tau}_{\text{diff}} \mid \mathbf{O}) = \frac{S_0^2}{n_0} + \frac{S_1^2}{n_1} - \frac{S_{\tau_i}^2}{n},$$

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$$S_{\tau_i}^2 = \frac{1}{n-1}\sum_{i=1}\left(Y_i(1) - Y_i(0) - \tau_{\text{fs}}\right)^2$$

- None of these are directly observable!

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	$i = 1$	$i = 2$	Avg.
$Y_i(0)$	10	-10	0
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τ_i	0	0	0

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- Upper bound that is only a function of identified parameters.

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- Can improve approximations using t-distribution.
    - Works since $\widehat{\mathbb{V}}$ will be approximately χ^2_{n-1} in large samples.

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  - Here, σ_0^2 and σ_1^2 are the population-level variances of $Y_i(1)$ and $Y_i(0)$.
- The variance of τ_i term drops out $\rightsquigarrow$ higher variance for PATE than SATE.

Estimating pop. sampling variance

$$\mathbb{V}(\widehat{\tau}_{\text{diff}}) = \frac{\sigma_0^2}{n_0} + \frac{\sigma_1^2}{n_1},$$

- Notice that the Neyman estimator $\widehat{\mathbb{V}}$ is now unbiased for $\mathbb{V}(\widehat{\tau}_{\text{diff}})$:

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- Iterated expectations gives us:

$$\tau_{\text{fs}} = \underbrace{\left(\frac{n_{\text{v}}}{n_{\text{v}} + n_{\text{nv}}}\right)}_{\text{fraction voters}} \tau_{\text{v, fs}} + \underbrace{\left(\frac{n_{\text{nv}}}{n_{\text{v}} + n_{\text{nv}}}\right)}_{\text{fraction nonvoters}} \tau_{\text{nv, fs}}$$

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Estimators in blocked designs

- Within-strata difference in means:

$$\widehat{\tau}_{\mathrm{v}} = \overline{Y}_{1,\mathrm{v}} - \overline{Y}_{0,\mathrm{v}} = \frac{1}{n_{1,\mathrm{v}}} \sum_{i:V_i=1} D_i Y_i - \frac{1}{n_{0,\mathrm{v}}} \sum_{i:V_i=1} (1 - D_i) Y_i$$

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  - Otherwise, standard $\widehat{\tau}_{\text{diff}}$ under block design will be **biased**.

Sampling variance of blocking estimator

- Each block is a completely randomized experiment so we have:

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  - Possible to get optimal blocks with **pair matching** ($J = n/2$).

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- Across-pair variance estimator (conservative for SATE):

$$\widehat{\mathbb{V}}(\widehat{\tau}_p) = \frac{1}{J(J-1)} \sum_{j=1}^{J} \{W_j (Y_{1j} - Y_{2j} - \widehat{\tau}_p)\}^2$$