Module 2: Randomization Inference

Fall 2021

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Gov 2003 (Harvard)

1/ Randomized experiments

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   - Sometimes written as $D_i \perp\!\!\!\perp (\mathbf{Y}(1), \mathbf{Y}(0))$

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- If information $\rightsquigarrow$ accountability, we should see a difference between the treatment and control groups.

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  - Important to admit limitations: external validity, sample selection, Hawthorne effect

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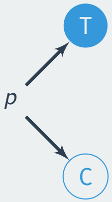
  - For any given i, implies $\mathbb{P}(D_i = 1 \mid \mathbf{Y}(1), \mathbf{Y}(0)) = \frac{n_1}{n}$.

Bernoulli assignment



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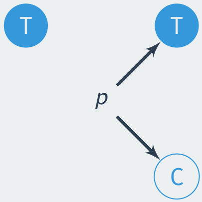

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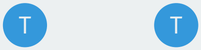

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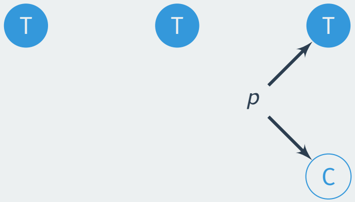

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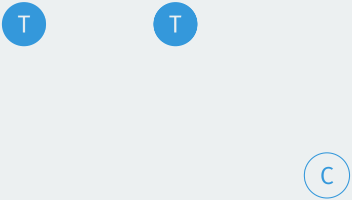

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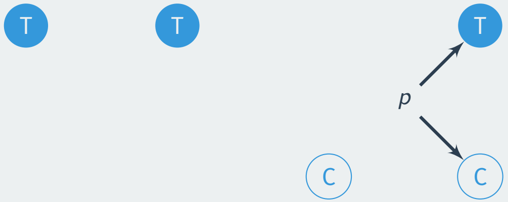

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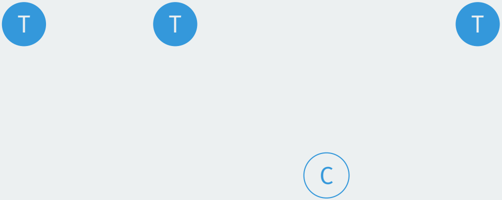

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  - When blocks are of size 2, this is a **pair-matched design**.

Example data from information RCT

Village	Information D_i	Incumbent Won? Y_i	$Y_i(0)$	$Y_i(1)$
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- Very small sample size $\rightsquigarrow$ can we learn anything from this data?

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    - $\rightsquigarrow$ truly nonparametric, but less flexible.

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1. Choose a null hypothesis:
    - $H_0 : \beta_1 = 0$ or $H_0 : \tau = 0$.
    - No average treatment effect.
    - Claim we would like to reject.
2. Choose a test statistic.
    - $Z_i = (X_i - \bar{X})/(s/\sqrt{n})$
3. Determine the distribution of the test statistic under the null.
    - Statistical thought experiment: we know the truth, what data should we expect?
4. Calculate the probability of the test statistics under the null.
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    - What is this called? **p-value**

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  - Here, $\tau = 0$ but the sharp null is violated.

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- What if treatment affected no one at all?
- Implies no **average** treatment effect, but no ATE $\not\Rightarrow$ sharp null.
  - Take a simple example with two units: $\tau_1 = 1 \qquad \tau_2 = -1$
  - Here, $\tau = 0$ but the sharp null is violated.
- If the sharp null is true, we know all the potential outcomes:

$$Y_i(1) = Y_i(0) = Y_i$$

Life under the sharp null

We can use the sharp null ($Y_i(1) - Y_i(0) = 0$) to fill in the missing potential outcomes:

	Information	Incumbent Won?		
Village	D_i	Y_i	$Y_i(0)$	$Y_i(1)$
1	1	0	?	0
2	1	0	?	0
3	0	1	1	?
4	1	0	?	0
5	1	1	?	1
6	0	1	1	?
7	0	0	0	?
8	1	1	?	1
9	0	1	1	?
10	0	0	0	?

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8	1	1	1	1
9	0	1	1	1
10	0	0	0	0

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- These will help us perform a test of the sharp null.
- Many possible tests to choose from!

Null/randomization disitribution

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- **Key insight of RI**: sharp null $\rightsquigarrow$ treatment assignment doesn't matter.
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  - $Y_i(1) = Y_i(0) = Y_i$
- **Randomization distribution**: distribution of T if the sharp null were true.

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    - p-values will be below α exactly $100\alpha\%$ of the time

Randomization inference step-by-step

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5. Repeat steps 3-4 for all Ω_0 to get $\widetilde{T} = \{\widetilde{T}_1, \dots, \widetilde{T}_K\}$.

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5. Repeat steps 3-4 for all Ω_0 to get $\widetilde{T} = \{\widetilde{T}_1, \ldots, \widetilde{T}_K\}$.
6. Calculate the p-value: $p = \frac{1}{K} \sum_{k=1}^{K} \mathbb{I}(\widetilde{T}_k \geq T)$

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- Larger values of T_{diff} are evidence against the sharp null.
- Good estimator for constant, additive treatment effects and relatively few outliers in the the potential outcomes.

Example

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- Afterwards, we observe them giving between $0 and $5.
- Simple example to show the steps of RI in a concrete case.

Randomization distribution

Unit	Mailer D_i	Contr. Y_i	$Y_i(0)$	$Y_i(1)$
Jon	1	3	(3)	3
Sansa	1	5	(5)	5
Arya	1	0	(0)	0
Robb	0	4	4	(4)
Bran	0	0	0	(0)
Rickon	0	1	1	(1)

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$$T_{\text{diff}} = |8/3 - 5/3| = 1$$

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$$\widetilde{T}_{\text{diff}} = |12/3 - 1/3| = 3.67$$

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$$\widetilde{T}_{\text{diff}} = |9/3 - 4/3| = 1.67$$

Randomization distribution

D_1	D_2	D_3	D_4	D_5	D_6	|Diff in means|

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In R

```
library(ri)
y <- c(3, 5, 0, 4, 0, 1)
D <- c(1, 1, 1, 0, 0, 0)
T_obs <- abs(mean(y[D == 1]) - mean(y[D == 0]))
D_bold <- ri::genperms(D)
D_bold[, 1:7]
```

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T_obs <- abs(mean(y[D == 1]) - mean(y[D == 0]))
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D_bold[, 1:7]
```

```
##   [,1] [,2] [,3] [,4] [,5] [,6] [,7]
## 1    1    1    1    1    1    1    1
## 2    1    1    1    1    0    0    0
## 3    1    0    0    0    1    1    1
## 4    0    1    0    0    1    0    0
## 5    0    0    1    0    0    1    0
## 6    0    0    0    1    0    0    1
```

Calculate means

```r
rdist <- rep(NA, times = ncol(D_bold))
for (i in seq_len(ncol(D_bold))) {
  D_tilde <- D_bold[, i]
  rdist[i] <- abs(mean(y[D_tilde == 1]) - mean(y[D_tilde == 0]))
}
rdist
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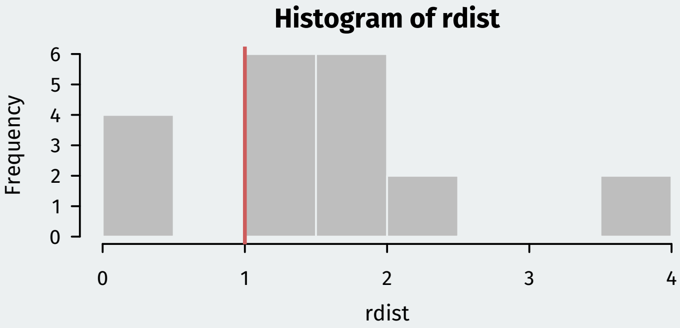
```
##  [1] 1.000 3.667 1.000 1.667 0.333 2.333 1.667 0.333
##  [9] 1.000 1.667 1.667 1.000 0.333 1.667 2.333 0.333
## [17] 1.667 1.000 3.667 1.000
```

p-value



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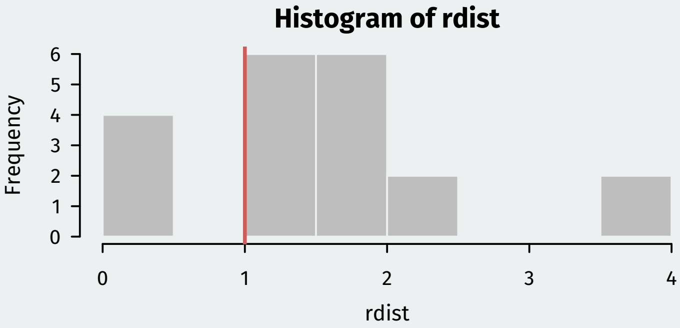


```
# p-value
mean(rdist >= T_obs)
```

```
## [1] 0.8
```

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- $n = 6$ and $n_1 = 3 \rightsquigarrow 20$ assignment vectors.

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- $n = 100$ and $n_1 = 50 \rightsquigarrow 1.009 \times 10^{29}$ vectors.

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  - calculate the randomization distribution in the K samples.

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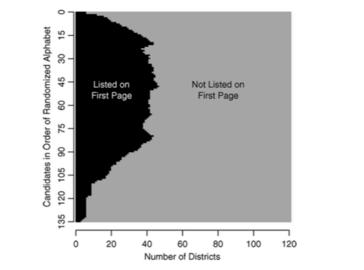
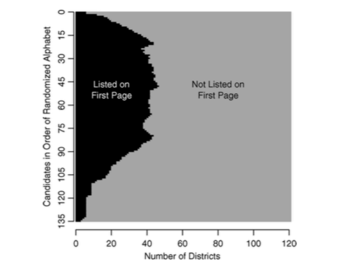
CA recall election



- Ho & Imai (2006): 135 candidates in 2003 CA Gov. recall election.

CA recall election



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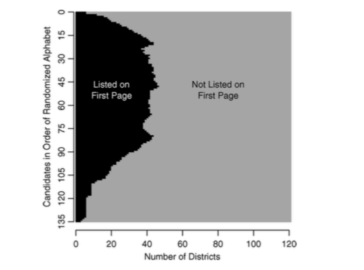

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- Useful for skewed distributions of outcomes.

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- Could use other quantiles (the 0.25 quantile or the 0.75 quantile).

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Rank statistics formally

- Calculate ranks of the outcomes:

$$\tilde{R}_i = \tilde{R}_i(Y_1, \dots, Y_n) = \sum_{j=1}^{N} \mathbb{I}(Y_j \leq Y_i)$$

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Randomization distribution

Unit	Mailer D_i	Contr. Y_i	$Y_i(0)$	$Y_i(1)$	Rank	R_i
Jon	1	3	(3)	3		
Sansa	1	5	(5)	5		
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$$T_{\text{rank}} = |1/3 - -1/3| = 0.67$$

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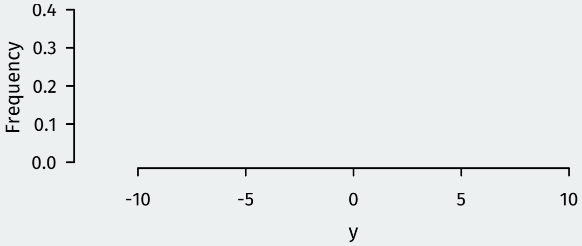

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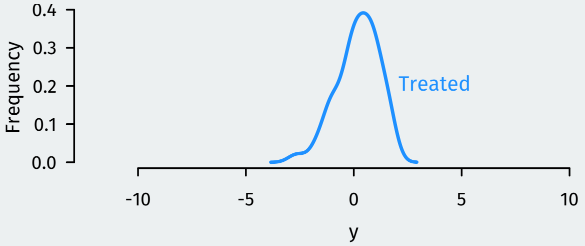

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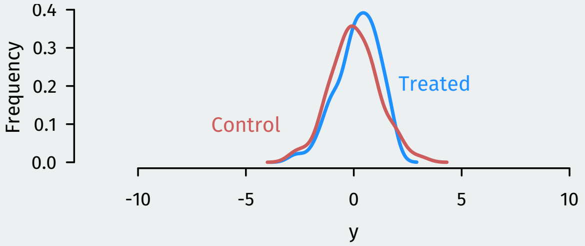

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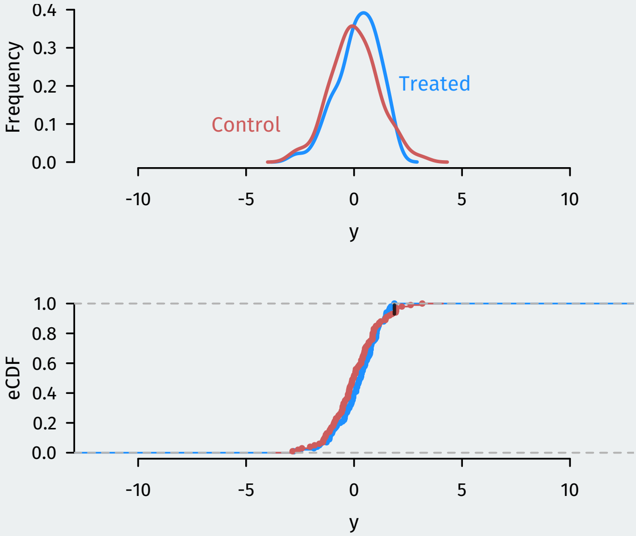

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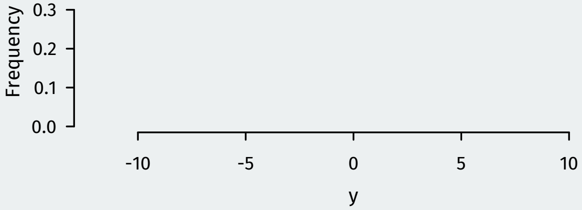

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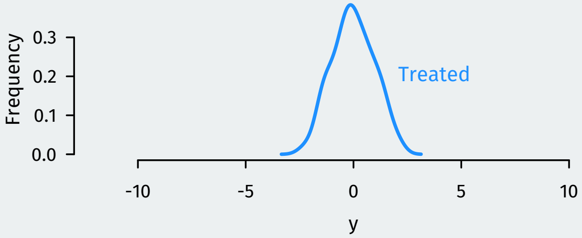

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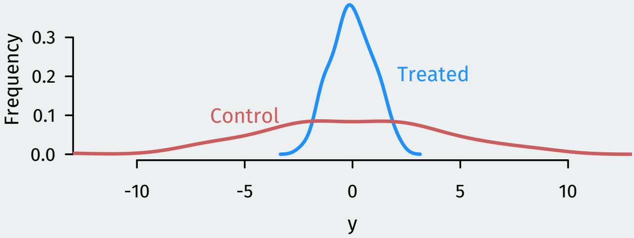

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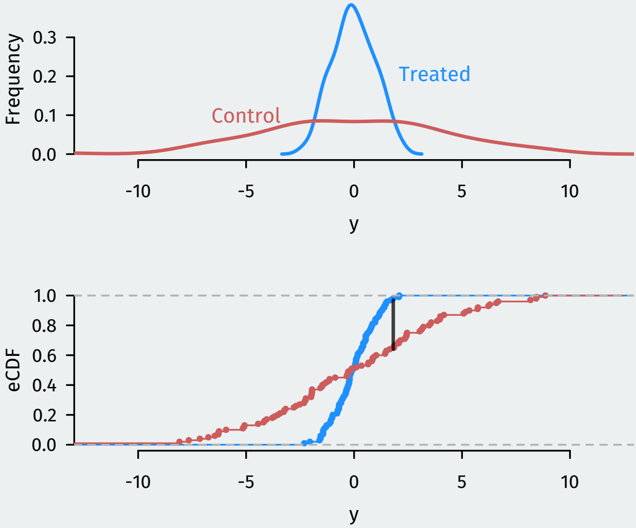

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- For these, use a test statistic that is bigger under the alternative:

$$T^*_{\text{diff}} = \bar{Y}_t - \bar{Y}_c$$

3/ Confidence intervals in randomization inference

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  - Collect all values that you cannot reject as the 95% CI.

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- Assignments will now affect Y_i.

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- If the p-value of 0.05 falls "between" two of these discrete points, a 95% CI will cover the true value more than 95% of the time.

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- Usually this means selecting the value with the highest p-value.

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- RI is justified **even if the model is wrong!**
    - OLS is just another way to generate a test statistic.
    - If the model is "right" (read: predictive of $Y_i(0)$), then T_{ols} will have higher power.