13. Properties of Least Squares

Fall 2023

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Gov 2002 (Harvard)

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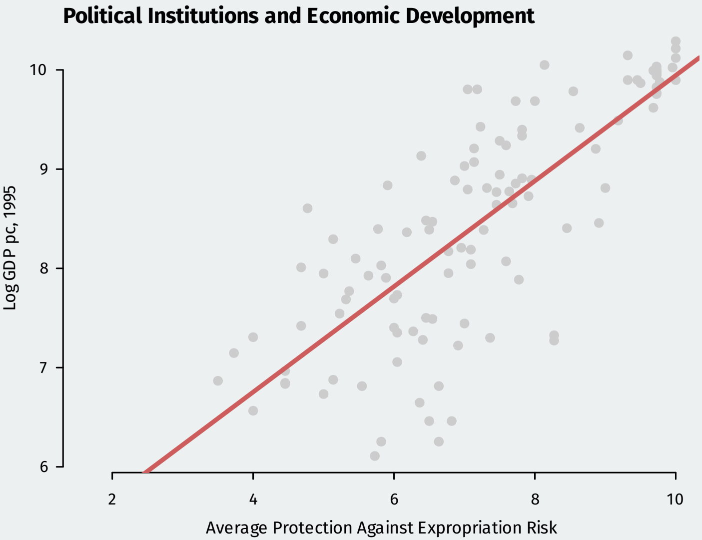
- Last time: OLS estimator, its algebraic properties.
- Now: its statistical properties, both finite-sample and asymptotic.

Acemoglu, Johnson, and Robinson (2001)



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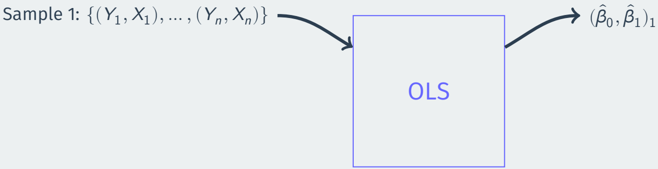

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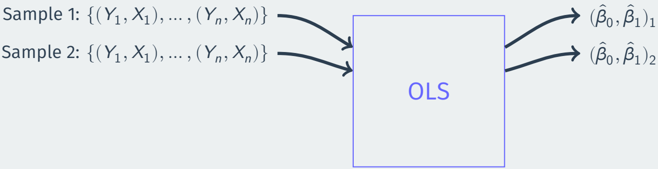

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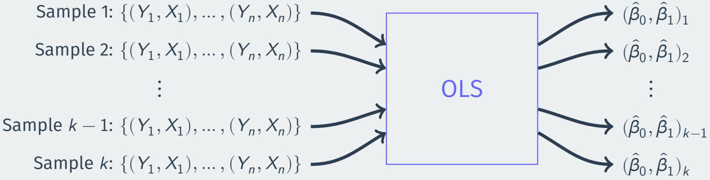

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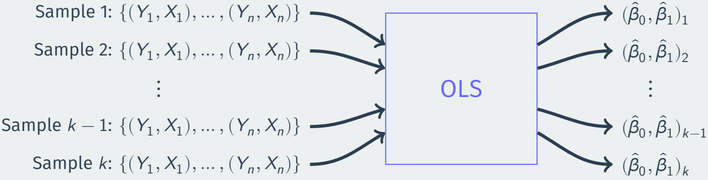

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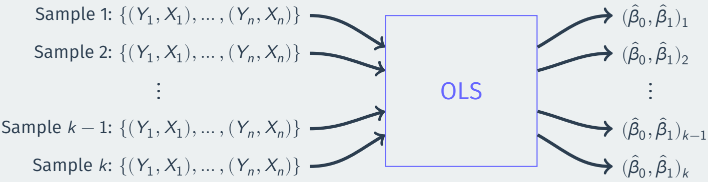

- Just like the sample mean or sample difference in means
- Has a sampling distribution, with a sampling variance/standard error.

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3. Plot the estimated regression line

Population Regression

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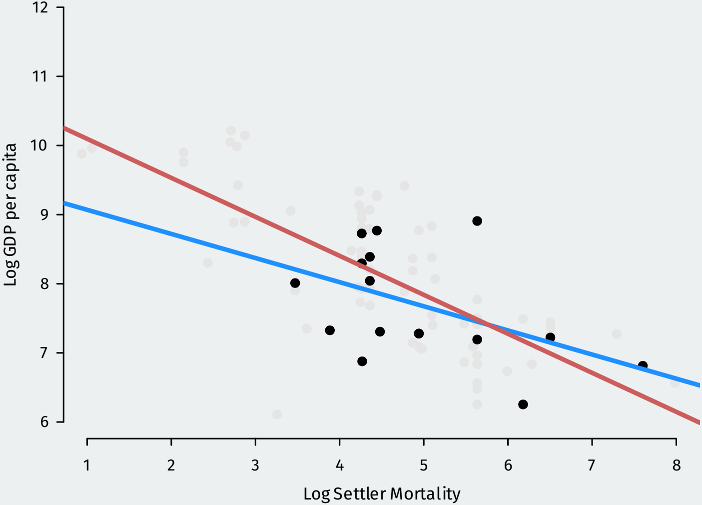

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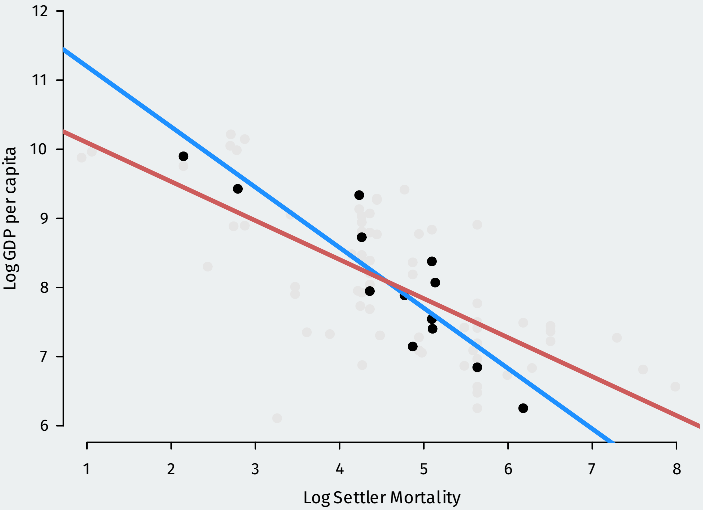

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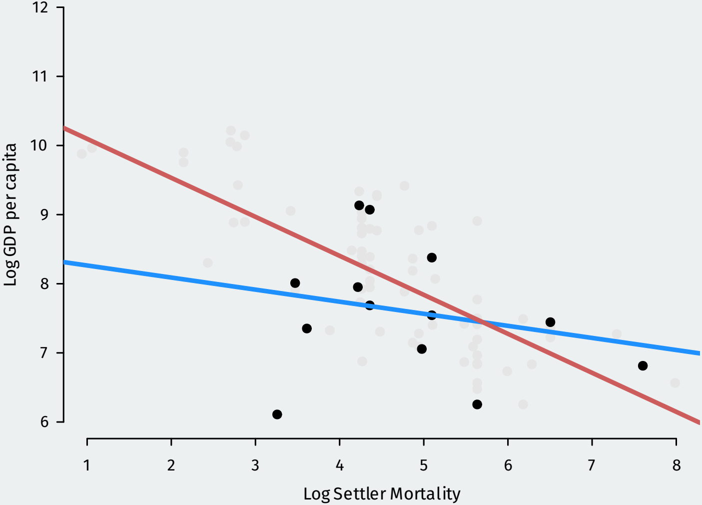

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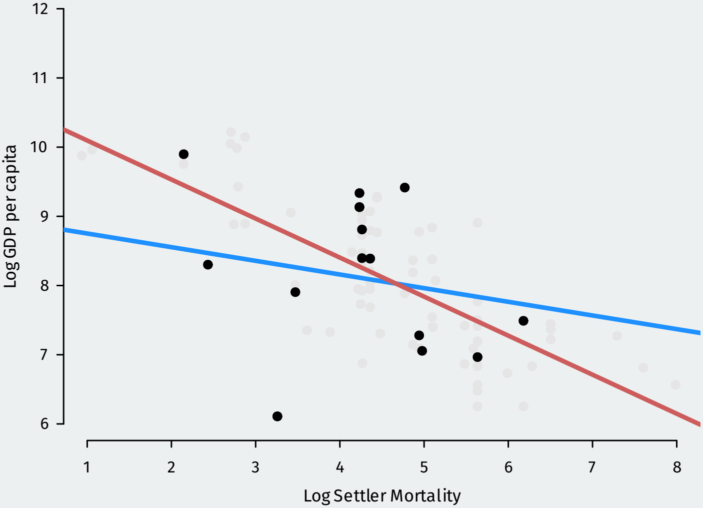

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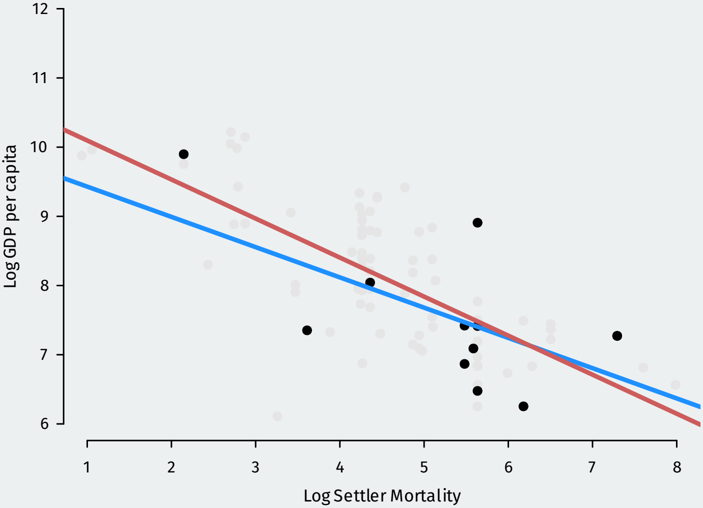

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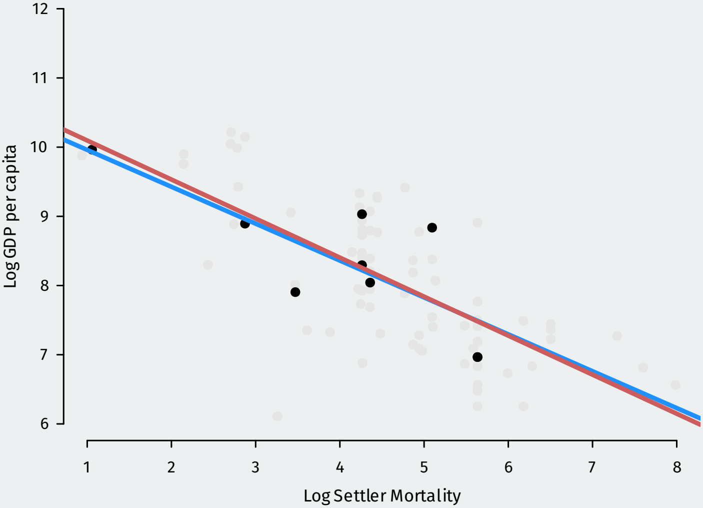

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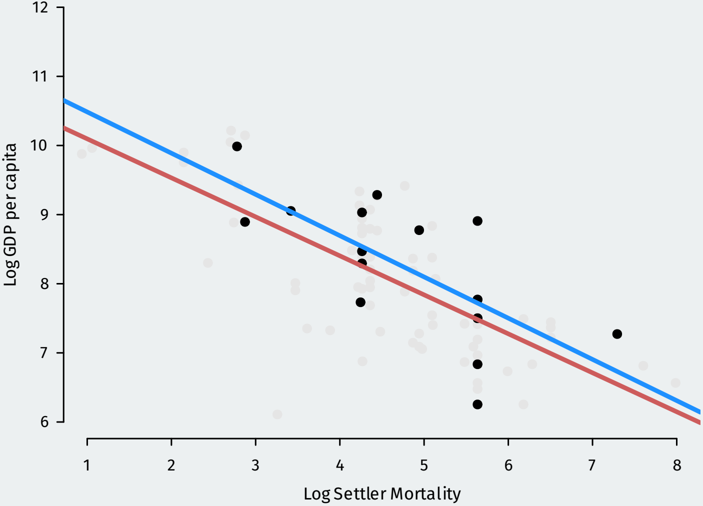

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    - **Linear regression/CEF model** for finite samples.

1/ Linear projection model and Large-sample Properties

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- What properties can we derive under such weak assumptions?

A very useful decomposition

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- $\mathbf{Q}_{\mathbf{XX}}$ is invertible by assumption, so by the continuous mapping theorem:

$$\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_i\mathbf{X}_i'\right)^{-1} \xrightarrow{p} \mathbf{Q}_{\mathbf{XX}}^{-1} \quad \Longrightarrow \quad \hat{\boldsymbol{\beta}} \xrightarrow{p} \boldsymbol{\beta} + \mathbf{Q}_{\mathbf{XX}}^{-1}\cdot\mathbf{0} = \boldsymbol{\beta},$$

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- Valid with no restrictions on Y: could be binary, discrete, etc.
- Not guaranteed to be unbiased (unless CEF is linear, as we'll see...)

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- If $\mathbb{E}[g(\mathbf{X}_i)] = 0$, then we have

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} g(\mathbf{X}_i)\right) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} g(\mathbf{X}_i) \xrightarrow{d} \mathcal{N}(0, \mathbb{E}[g(\mathbf{X}_i)g(\mathbf{X}_i)'])$$

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    - Rewrite as $\sqrt{n}$ times an average of i.i.d. mean-zero random vectors.

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    - $n^{-1}\sum_{i=1}^{n}\mathbf{X}_i e_i$ is a sample average with $\mathbb{E}[\mathbf{X}_i e_i] = 0$.
    - Rewrite as $\sqrt{n}$ times an average of i.i.d. mean-zero random vectors.
- Let $\boldsymbol{\Omega} = \mathbb{E}[e_i^2\mathbf{X}_i\mathbf{X}_i']$ and apply the CLT:

$$\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{X}_i e_i\right) \xrightarrow{d} \mathcal{N}(0, \boldsymbol{\Omega})$$

Asymptotic normality

Theorem (Asymptotic Normality of OLS)

Under the linear projection model,

$$\sqrt{n}\left(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) \xrightarrow{d} \mathcal{N}(0, \mathbf{V}_{\boldsymbol{\beta}}),$$

where,

$$\mathbf{V}_{\boldsymbol{\beta}} = \mathbf{Q}_{\mathbf{XX}}^{-1}\boldsymbol{\Omega}\mathbf{Q}_{\mathbf{XX}}^{-1} = \left(\mathbb{E}[\mathbf{X}_i\mathbf{X}_i']\right)^{-1}\mathbb{E}[e_i^2\mathbf{X}_i\mathbf{X}_i']\left(\mathbb{E}[\mathbf{X}_i\mathbf{X}_i']\right)^{-1}$$

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    - Square root of the diagonal of $\mathbf{V}_{\hat{\boldsymbol{\beta}}}$ = standard errors for $\hat{\beta}_j$
- Allows us to formulate (approximate) confidence intervals, tests.

2/ OLS variance estimation

Estimating OLS variance

$$\sqrt{n}\left(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) \xrightarrow{d} \mathcal{N}(0, \mathbf{V}_{\boldsymbol{\beta}}), \qquad \mathbf{V}_{\boldsymbol{\beta}} = \mathbf{Q}_{\mathbf{XX}}^{-1} \boldsymbol{\Omega} \mathbf{Q}_{\mathbf{XX}}^{-1}$$

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$$\widehat{\mathbf{V}}_{\boldsymbol{\beta}} = \left(\frac{1}{n}\mathbb{X}'\mathbb{X}\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^n \hat{e}_i^2\mathbf{X}_i\mathbf{X}_i'\right)\left(\frac{1}{n}\mathbb{X}'\mathbb{X}\right)^{-1} \xrightarrow{p} \mathbf{V}_{\boldsymbol{\beta}}$$

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- Approximate variance of the coefficients:

$$\widehat{\mathbf{V}}_{\hat{\boldsymbol{\beta}}} = \frac{1}{n}\widehat{\mathbf{V}}_{\boldsymbol{\beta}} = (\mathbb{X}'\mathbb{X})^{-1}\left(\sum_{i=1}^{n}\hat{e}_i^2\mathbf{X}_i\mathbf{X}_i'\right)(\mathbb{X}'\mathbb{X})^{-1}$$

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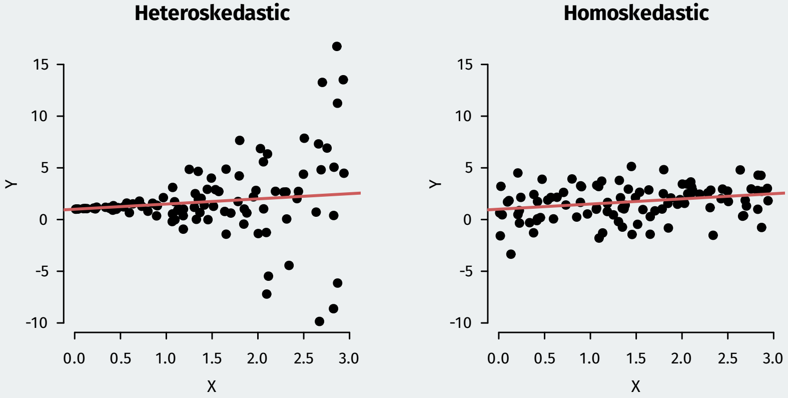
- Square root of the diagonal of $\widehat{\mathbf{V}}_{\hat{\boldsymbol{\beta}}}$: **heteroskedasticity-consistent (HC) SEs** (aka “robust SEs”)

Homoskedasticity

Assumption: Homoskedasticity

The variance of the error terms is constant in $\mathbf{X}$, $\mathbb{E}[e^2 \mid \mathbf{X}] = \sigma^2(\mathbf{X}) = \sigma^2$.



Consequences of homoskedasticity

- Homoskedasticity implies $\mathbb{E}[e_i^2 \mathbf{X}_i \mathbf{X}_i'] = \mathbb{E}[e_i^2]\mathbb{E}[\mathbf{X}_i \mathbf{X}_i'] = \sigma^2 \mathbf{Q}_{\mathbf{XX}}$

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- Estimated variance of $\hat{\boldsymbol{\beta}}$ under homoskedasticity

$$s^2 = \frac{1}{n-k} \sum_{i=1}^{n} \hat{e}_i^2 \qquad \widehat{\mathbf{V}}_{\hat{\boldsymbol{\beta}}}^{\texttt{lm}} = \frac{1}{n} s^2 \left(\frac{1}{n} \sum_{i=1}^{n} \mathbf{X}_i \mathbf{X}_i' \right)^{-1} = s^2 \left(\mathbb{X}' \mathbb{X} \right)^{-1}$$

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- LLN implies $s^2 \overset{p}{\to} \sigma^2$ and so $n \widehat{\mathbf{V}}_{\hat{\boldsymbol{\beta}}}^{\texttt{lm}}$ is consistent for $\mathbf{V}_{\boldsymbol{\beta}}^{\texttt{lm}}$

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    - Mostly small, ad hoc changes to improve finite-sample performance.

AJR data

```
library(sandwich)
mod <- lm(logpgp95 ~ avexpr + lat_abst + meantemp, data = ajr)
vcov(mod) ## homoskdastic V_\hat{beta}
```

```
##             (Intercept)    avexpr  lat_abst  meantemp
## (Intercept)      0.9079 -0.040952 -0.537463 -0.023246
## avexpr          -0.0410  0.004162 -0.000778  0.000605
## lat_abst        -0.5375 -0.000778  0.867588  0.016717
## meantemp        -0.0232  0.000605  0.016717  0.000705
```

```
sandwich::vcovHC(mod, type = "HC2") ## HC2
```

```
##             (Intercept)   avexpr lat_abst  meantemp
## (Intercept)      0.9764 -0.05735 -0.29548 -0.024639
## avexpr          -0.0573  0.00538 -0.00358  0.001107
## lat_abst        -0.2955 -0.00358  0.60821  0.008792
## meantemp        -0.0246  0.00111  0.00879  0.000706
```

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$$\text{general t-statistic} = \frac{\hat{\beta}_j - b_0}{\widehat{\text{se}}(\hat{\beta}_j)} \qquad \text{"usual" t-statistic} = \frac{\hat{\beta}_j}{\widehat{\text{se}}(\hat{\beta}_j)}$$

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- Software often uses t critical values instead of normal (we'll see why).

Inference with `lmtest::coeftest()`

```r
library(lmtest)
## homoskedastic error
lmtest::coeftest(mod)
```

```
##
## t test of coefficients:
##
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)   6.9289     0.9528    7.27  1.2e-09 ***
## avexpr        0.4059     0.0645    6.29  5.1e-08 ***
## lat_abst     -0.1980     0.9314   -0.21    0.832
## meantemp     -0.0641     0.0266   -2.41    0.019 *
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```r
## HC2 variance estimator
lmtest::coeftest(mod, vcov = vcovHC(mod, type = "HC2"))
```

```
##
## t test of coefficients:
##
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)   6.9289     0.9881    7.01  3.3e-09 ***
## avexpr        0.4059     0.0733    5.53  8.6e-07 ***
## lat_abst     -0.1980     0.7799   -0.25    0.801
## meantemp     -0.0641     0.0266   -2.41    0.019 *
## ---
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3/ Inference for Multiple Parameters

Inference for interactions

$$m(x, z) = \beta_0 + X\beta_1 + Z\beta_2 + XZ\beta_3$$

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- What if we want the variance of this effect for any value of Z?

$$\mathbb{V}\left(\frac{\partial \widehat{m}(x,z)}{\partial x}\right) = \mathbb{V}\left[\hat{\beta}_1 + z\hat{\beta}_3\right] = \mathbb{V}[\hat{\beta}_1] + z^2\mathbb{V}[\hat{\beta}_3] + 2z\text{cov}[\hat{\beta}_1, \hat{\beta}_3]$$

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- Use the estimated covariance matrix:

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Inference for interactions

$$m(x, z) = \beta_0 + X\beta_1 + Z\beta_2 + XZ\beta_3$$

- **Partial** or **marginal** effect of X at Z: $\frac{\partial m(x,z)}{\partial x} = \beta_1 + z\beta_3$
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  - $\widehat{V}_{\widehat{\beta}_1}$ is the diagonal entry of $\widehat{\mathbf{V}}_{\widehat{\boldsymbol{\beta}}}$ for $\widehat{\beta}_1$

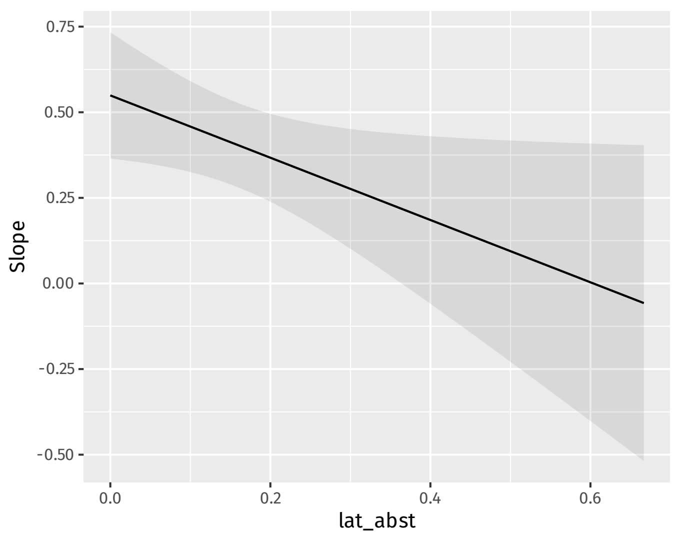
Visualizing via `marginaleffects`

```r
int_mod <- lm(logpgp95 ~ avexpr * lat_abst + meantemp, data = ajr)
coeftest(int_mod)
```

```
##
## t test of coefficients:
##
##                 Estimate Std. Error t value Pr(>|t|)
## (Intercept)       6.9864     0.9273    7.53    5e-10
## avexpr            0.5491     0.0941    5.84    3e-07
## lat_abst          5.8152     3.0791    1.89   0.0642
## meantemp         -0.1048     0.0326   -3.21   0.0022
## avexpr:lat_abst  -0.9095     0.4451   -2.04   0.0458
##
## (Intercept)     ***
## avexpr          ***
## lat_abst        .
## meantemp        **
## avexpr:lat_abst *
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Visualizing marginal effects

```
library(marginaleffects)
plot_slopes(int_mod, variables = "avexpr", condition = "lat_abst")
```



Tests of multiple coefficients

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    - Need to normalize like the t-statistic.

Alternative test for one coefficient

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Rewriting hypotheses with matrices

- We can rewrite the null hypothesis as $H_0 : \mathbf{L}\boldsymbol{\beta} = \mathbf{c}$ where,

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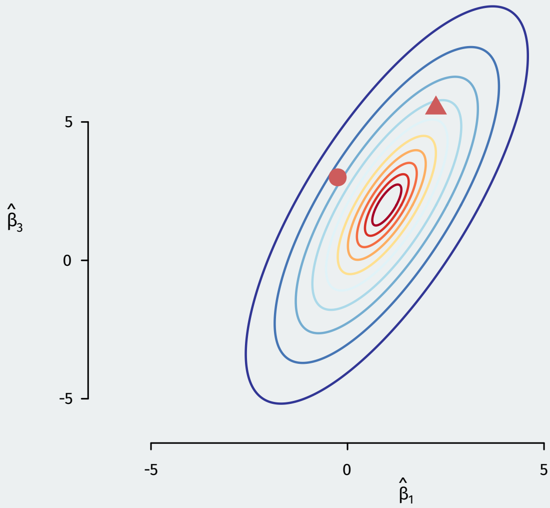
  - Squared distance of observed values from the null, weighted by the distribution of the parameters under the null

Weighting by the distribution



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  - "Usual" F-test reports test of all coef = 0 except intercept (pointless?)

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- When applied to a single coefficient, equivalent to a t-test.

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    - We can get this from R using the `qchisq()` function
3. Reject if observed statistic is bigger than critical value
    - Use `pchisq()` to get p-values if needed.

- When applied to a single coefficient, equivalent to a t-test.
- Use packages like `{lmtest}` or `{clubSandwich}` in R.

Wald test in lmtest

```r
## run OLS with the restrictions imposed (avexpr removed)
restricted <- lm(logpgp95 ~ lat_abst + meantemp, data = ajr)

## pass estimated model and estimated null model to
## wald test with HC variance estimator
lmtest::waldtest(restricted, int_mod, test = "Chisq",
                 vcov = vcovHC)
```

```
## Wald test
##
## Model 1: logpgp95 ~ lat_abst + meantemp
## Model 2: logpgp95 ~ avexpr * lat_abst + meantemp
##   Res.Df Df Chisq Pr(>Chisq)
## 1     57
## 2     55  2  34.2    3.7e-08 ***
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

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- Illustration:
    - Randomly draw 21 variables independently.
    - Run a regression of the first variable on the rest.
- By design, no effect of any variable on any other.

Multiple test example

```
noise <- data.frame(matrix(rnorm(2100), nrow = 100, ncol = 21))
summary(lm(noise))
```

```
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) -0.028039   0.113820   -0.25   0.8061
## X2          -0.150390   0.112181   -1.34   0.1839
## X3           0.079158   0.095028    0.83   0.4074
## X4          -0.071742   0.104579   -0.69   0.4947
## X5           0.172078   0.114002    1.51   0.1352
## X6           0.080852   0.108341    0.75   0.4577
## X7           0.102913   0.114156    0.90   0.3701
## X8          -0.321053   0.120673   -2.66   0.0094 **
## X9          -0.053122   0.107983   -0.49   0.6241
## X10          0.180105   0.126443    1.42   0.1583
## X11          0.166386   0.110947    1.50   0.1377
## X12          0.008011   0.103766    0.08   0.9387
## X13          0.000212   0.103785    0.00   0.9984
## X14         -0.065969   0.112214   -0.59   0.5583
## X15         -0.129654   0.111575   -1.16   0.2487
## X16         -0.054446   0.125140   -0.44   0.6647
## X17          0.004335   0.112012    0.04   0.9692
## X18         -0.080796   0.109853   -0.74   0.4642
## X19         -0.085806   0.118553   -0.72   0.4713
## X20         -0.186006   0.104560   -1.78   0.0791 .
## X21          0.002111   0.108118    0.02   0.9845
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.999 on 79 degrees of freedom
## Multiple R-squared:  0.201,  Adjusted R-squared:  -0.00142
## F-statistic: 0.993 on 20 and 79 DF,  p-value: 0.48
```

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  - Example: $0.05/20 = 0.0025$
  - Ensures that the family-wise error rate (probability of making at least 1 Type I error) is less than α.

4/ Linear Regression Model and Finite-sample Properties

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- We continue to maintain $\{(Y_i, \mathbf{X}_i)\}$ are i.i.d.

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- Useful when linearity holds by default (discrete X in experiments, etc)

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  - A matrix $\mathbf{C}$ is p.s.d. if $\mathbf{x}'\mathbf{C}\mathbf{x} \geq 0$.
  - Upshot: OLS will have the smaller SEs than any other linear estimator.

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- With reasonable n, asymptotic normality has the same effect.

5/ Clustering

Clustered dependence: intuition

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- Called **clustering** or **clustered dependence**

Clustered dependence: notation

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    - rulings in judges
- Outcome varies at the unit-level, Y_{ig} and the main independent variable varies at the cluster level, X_g.

Clustered dependence: example model

$$
\begin{aligned}
Y_{ig} &= \beta_0 + X_g\beta_1 + v_{ig} \\
&= \beta_0 + X_g\beta_1 + c_g + u_{ig}
\end{aligned}
$$

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- c_g and u_{ig} are assumed to be independent of each other.
    - $\rightsquigarrow \mathbb{V}[v_{ig}|X_g] = \sigma_c^2 + \sigma_u^2$
- What if we ignore this structure and just use v_{ig} as the error?

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- Zero covariance of two units i and s in different clusters g and k:

$$\text{Cov}[v_{ig}, v_{sk}] = 0$$

Example covariance matrix

- $\mathbf{v}' = \begin{bmatrix} v_{1,1} & v_{2,1} & v_{3,1} & v_{4,2} & v_{5,2} & v_{6,2} \end{bmatrix}$

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- Variance matrix under i.i.d.:

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- We can write the OLS estimator as:

$$\widehat{\boldsymbol{\beta}} = \left(\sum_{g=1}^{m} \mathbb{X}_g'\mathbb{X}_g\right)\left(\sum_{g=1}^{m} \mathbb{X}_g'\mathbf{Y}_g\right)$$

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$$\widehat{\mathbf{V}}_{\hat{\boldsymbol{\beta}}}^{\text{CL0}} = (\mathbb{X}'\mathbb{X})^{-1} \left(\sum_{g=1}^{m} \mathbb{X}_g'\hat{\mathbf{v}}_g\hat{\mathbf{v}}_g'\mathbb{X}_g\right) (\mathbb{X}'\mathbb{X})^{-1}$$

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- With small-sample adjustment (reported by most software):

$$\widehat{\mathbf{V}}_{\hat{\boldsymbol{\beta}}}^{\text{CL1}} = \frac{m}{m-1}\frac{n-1}{n-k}(\mathbb{X}'\mathbb{X})^{-1}\left(\sum_{g=1}^{m}\mathbb{X}_g'\hat{\mathbf{v}}_g\hat{\mathbf{v}}_g'\mathbb{X}_g\right)(\mathbb{X}'\mathbb{X})^{-1}$$

Example: Gerber, Green, Larimer

Dear Registered Voter:

WHAT IF YOUR NEIGHBORS KNEW WHETHER YOU VOTED?

Why do so many people fail to vote? We've been talking about the problem for years, but it only seems to get worse. This year, we're taking a new approach. We're sending this mailing to you and your neighbors to publicize who does and does not vote.

The chart shows the names of some of your neighbors, showing which have voted in the past. After the August 8 election, we intend to mail an updated chart. You and your neighbors will all know who voted and who did not.

DO YOUR CIVIC DUTY — VOTE!

MAPLE DR	Aug 04	Nov 04	Aug 06
9995 JOSEPH JAMES SMITH	Voted	Voted	______
9995 JENNIFER KAY SMITH		Voted	______
9997 RICHARD B JACKSON		Voted	______
9999 KATHY MARIE JACKSON		Voted	______

Social pressure model

```
load("../assets/gerber_green_larimer.RData")
library(lmtest)
social$voted <- 1 * (social$voted == "Yes")
social$treatment <- factor(
  social$treatment,
  levels = c("Control", "Hawthorne", "Civic Duty", "Neighbors", "Self")
)
mod1 <- lm(voted ~ treatment, data = social)
coeftest(mod1)
```

```
##
## t test of coefficients:
##
##                       Estimate Std. Error t value
## (Intercept)            0.29664    0.00106  279.53
## treatmentHawthorne     0.02574    0.00260    9.90
## treatmentCivic Duty    0.01790    0.00260    6.88
## treatmentNeighbors     0.08131    0.00260   31.26
## treatmentSelf          0.04851    0.00260   18.66
##                       Pr(>|t|)
## (Intercept)            < 2e-16 ***
## treatmentHawthorne     < 2e-16 ***
## treatmentCivic Duty    5.8e-12 ***
## treatmentNeighbors     < 2e-16 ***
## treatmentSelf          < 2e-16 ***
## ---
```

Social pressure model, CRSEs

```r
library(sandwich)
coeftest(mod1, vcov = sandwich::vcovCL(mod1, cluster = social$hh_id))
```

```
##
## t test of coefficients:
##
##                      Estimate Std. Error t value
## (Intercept)           0.29664    0.00131  226.52
## treatmentHawthorne    0.02574    0.00326    7.90
## treatmentCivic Duty   0.01790    0.00324    5.53
## treatmentNeighbors    0.08131    0.00337   24.13
## treatmentSelf         0.04851    0.00330   14.70
##                      Pr(>|t|)
## (Intercept)           < 2e-16 ***
## treatmentHawthorne    2.8e-15 ***
## treatmentCivic Duty   3.2e-08 ***
## treatmentNeighbors    < 2e-16 ***
## treatmentSelf         < 2e-16 ***
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

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  - CRSEs can be incorrect with a small (< 50 maybe) number of clusters