12. Algebra of Least Squares

Fall 2023

Matthew Blackwell

Gov 2002 (Harvard)

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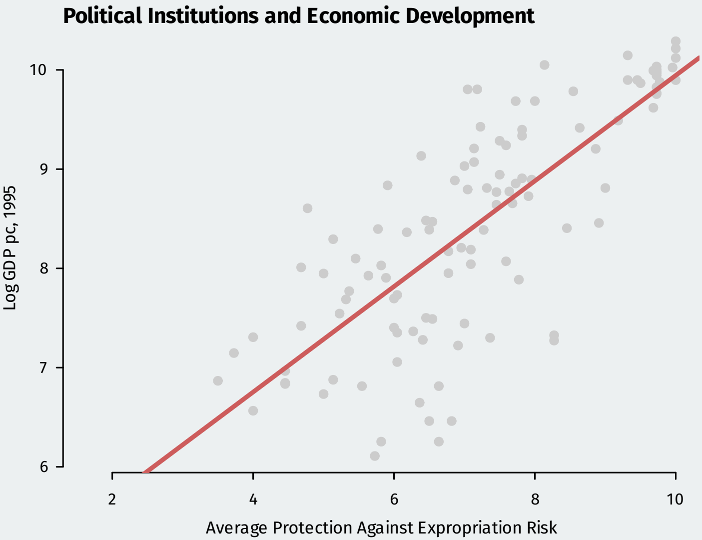
- Now: least squares estimator and its algebraic properties.
- After that: the statistical properties of least squares.

Acemoglu, Johnson, and Robinson (2001)



1/ Deriving the OLS estimator

Samples vs population

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- Violations include time-series data and clustered sampling.
  - Weakening i.i.d. usually complicates notation but can be done.

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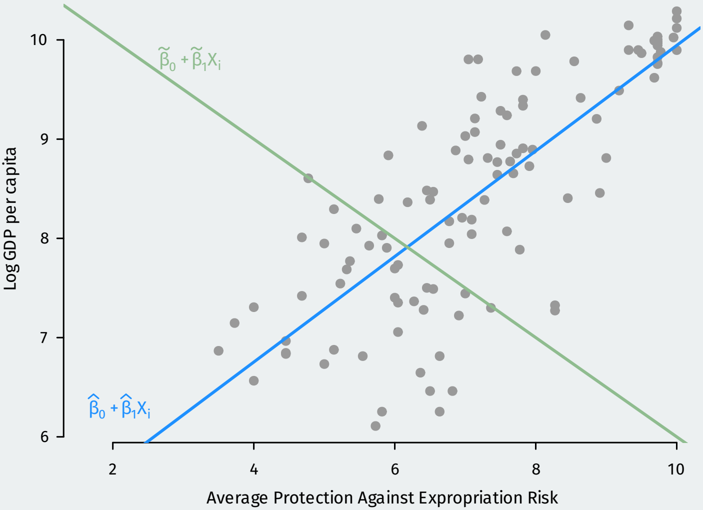

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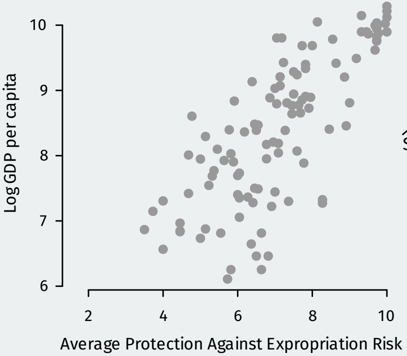
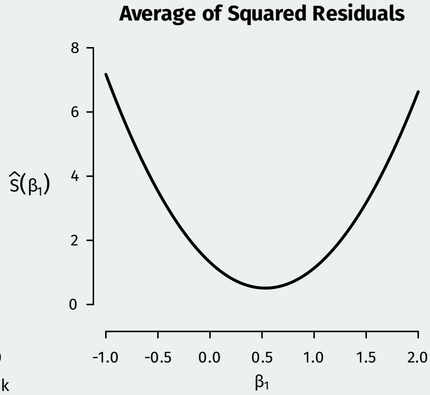
- We can show the OLS estimator of the slope is:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(Y_i - \overline{Y})(X_i - \overline{X})}{\sum_{i=1}^{n}(X_i - \overline{X})^2} = \frac{\widehat{\text{Cov}}(X, Y)}{\hat{\mathbb{V}}[X]}$$

Visualizing OLS





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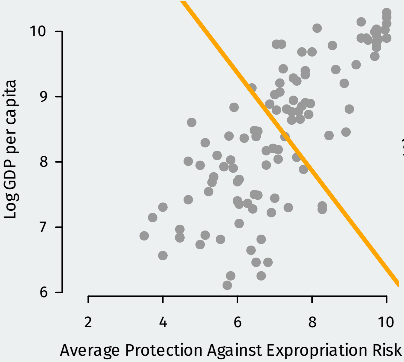

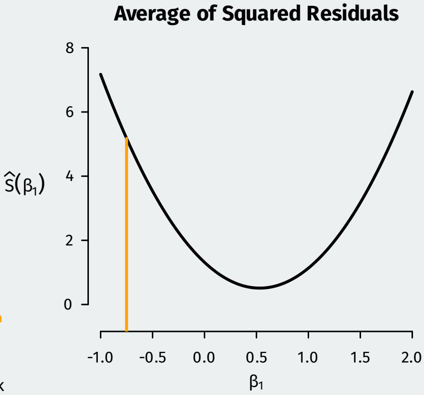

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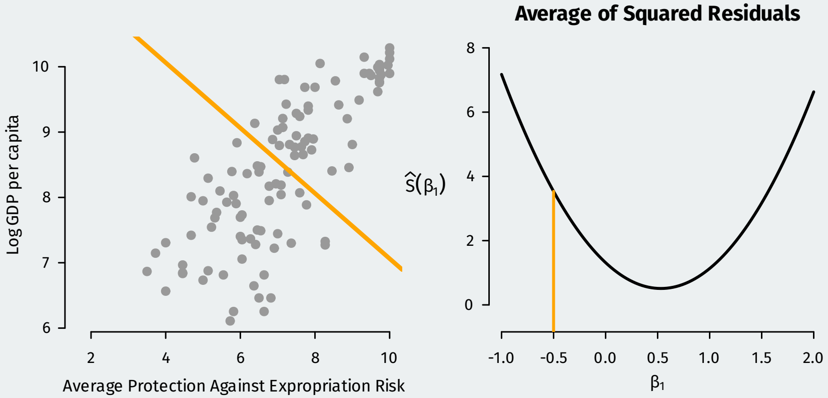

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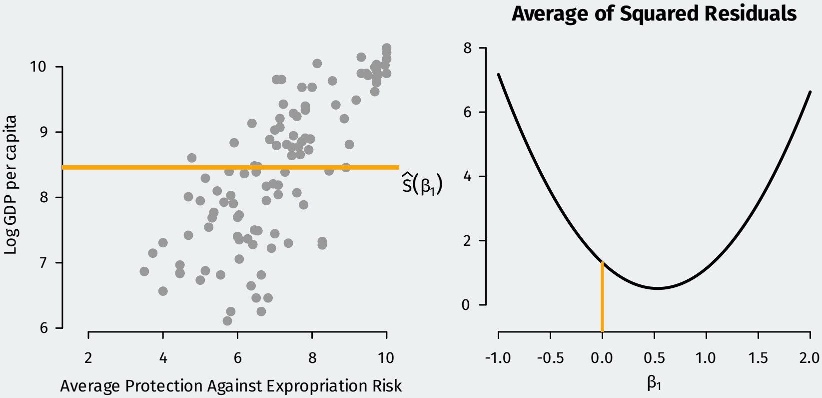

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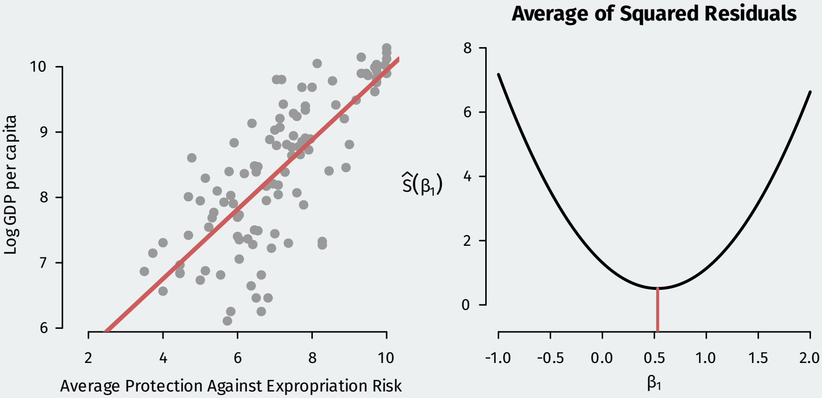

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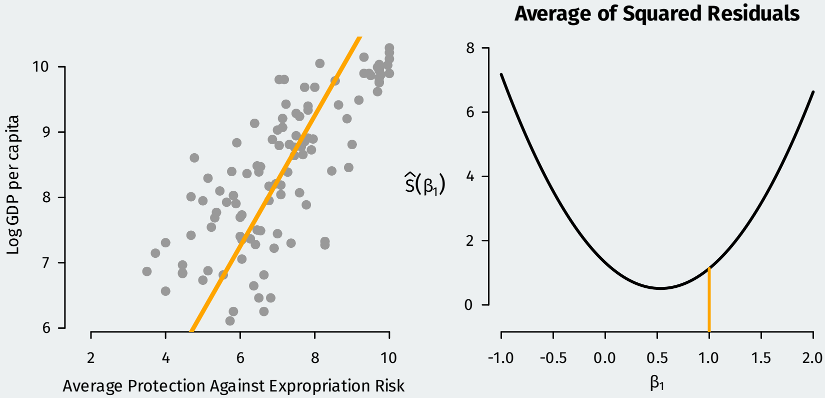

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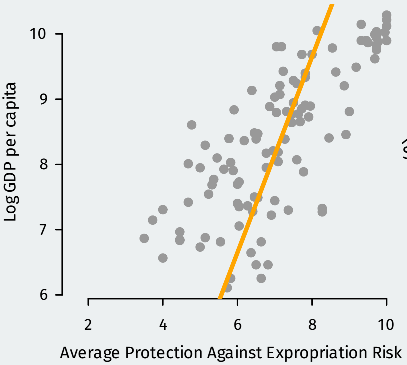

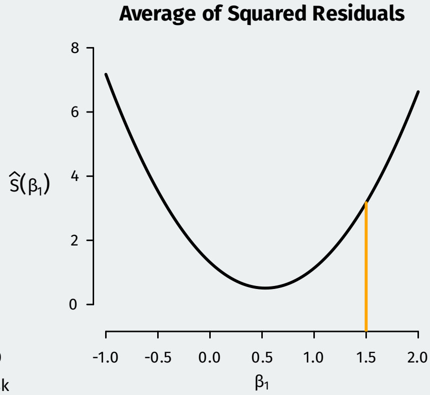

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2/ Model fit

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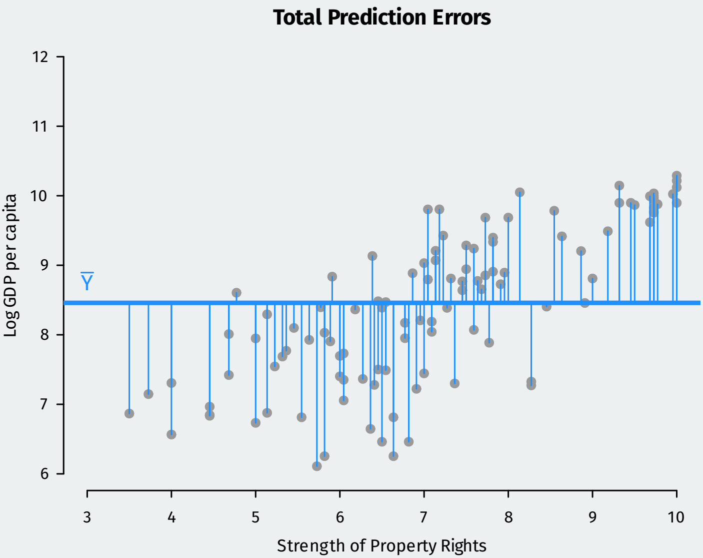
Total SS vs SSR



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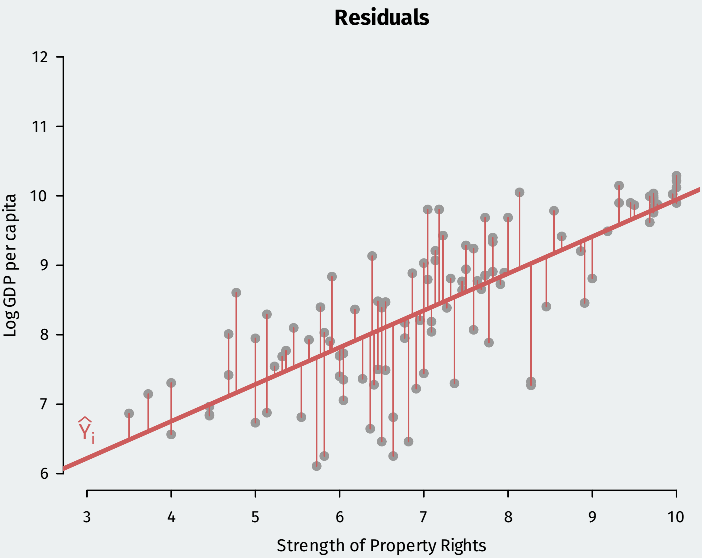

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- **Common interpretation:** R^2 is the fraction of the variation in Y_i is "explained by" $\mathbf{X}_i$.

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- $R^2 =$ fraction of the total prediction error eliminated by using $\mathbf{X}_i$.
- **Common interpretation:** R^2 is the fraction of the variation in Y_i is "explained by" $\mathbf{X}_i$.
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- Mechanically increases with additional covariates (better fit measures exist)

3/ Geometry of OLS

Linear model in matrix form

- Linear model is a system of n linear equations:

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\begin{aligned}
Y_1 &= \mathbf{X}_1'\boldsymbol{\beta} + e_1 \\
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\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix}, \quad \mathbb{X} = \begin{pmatrix} \mathbf{X}_1' \\ \mathbf{X}_2' \\ \vdots \\ \mathbf{X}_n' \end{pmatrix} = \begin{pmatrix} 1 & X_{11} & X_{12} & \cdots & X_{1k} \\ 1 & X_{21} & X_{22} & \cdots & X_{2k} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & X_{n1} & X_{n2} & \cdots & X_{nk} \end{pmatrix}, \quad \mathbf{e} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}
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- Model is now just:

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\mathbf{Y} = \mathbb{X}\boldsymbol{\beta} + \mathbf{e}
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OLS estimator in matrix form

- Key relationship: sample sums can be written in matrix notation:

$$\sum_{i=1}^{n} \mathbf{X}_i \mathbf{X}_i' = \mathbb{X}'\mathbb{X}$$

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    - Projecting $\mathbb{X}$ onto itself returns itself: $\mathbf{P}\mathbb{X} = \mathbb{X}$

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- Allows the following orthogonal partition:

$$\mathbf{Y} = \mathbf{PY} + \mathbf{MY} = \text{projection} + \text{residual}$$

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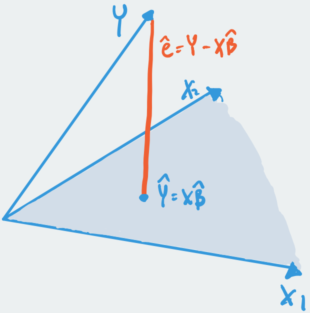
    - Find the point in $\mathcal{C}(\mathbb{X})$ that is closest to $\mathbf{Y}$

Projection



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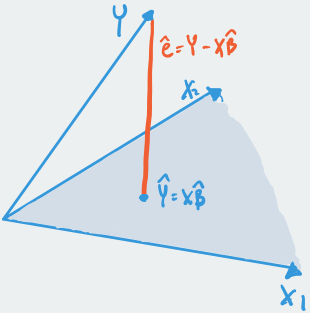

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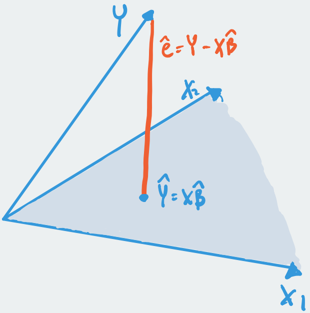

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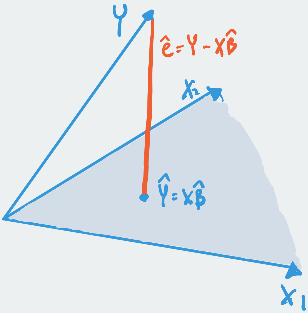

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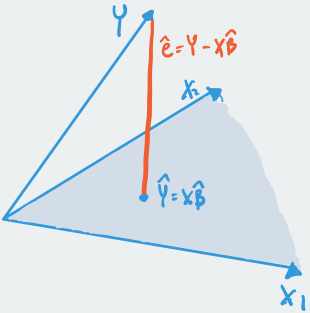
Projection



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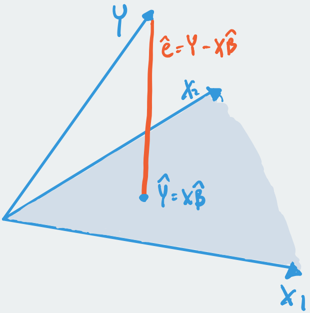

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4/ Partitioned regression and partial regression

Partitioned regression

- Partition covariates and coefficients $\mathbb{X} = [\mathbb{X}_1 \ \mathbb{X}_2]$ and $\boldsymbol{\beta} = (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)'$:

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- With exactly orthogonal covariates, multivariate OLS is the same as univariate OLS.
- Only holds in balanced, designed experiments.

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- Consider the OLS slope with an intercept:

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  2. Regress $\mathbf{Y}$ on residual from

Visualizing orthogonalization

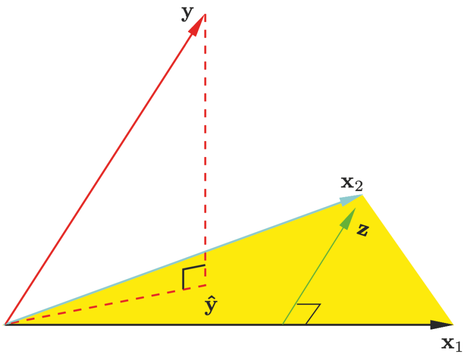

FIGURE 3.4. *Least squares regression by orthogonalization of the inputs. The vector* $\mathbf{x}_2$ *is regressed on the vector* $\mathbf{x}_1$*, leaving the residual vector* $\mathbf{z}$*. The regression of* $\mathbf{y}$ *on* $\mathbf{z}$ *gives the multiple regression coefficient of* $\mathbf{x}_2$*. Adding together the projections of* $\mathbf{y}$ *on each of* $\mathbf{x}_1$ *and* $\mathbf{z}$ *gives the least squares fit* $\hat{\mathbf{y}}$*.*

Why does residual regression work?

- We can find $\hat{\boldsymbol{\beta}}_1$ by nested minimization:

$$\hat{\boldsymbol{\beta}}_1 = \arg\min_{\boldsymbol{\beta}_1} \left(\min_{\boldsymbol{\beta}_2} \|\mathbf{Y} - \mathbb{X}_1\boldsymbol{\beta}_1 - \mathbb{X}_2\boldsymbol{\beta}_2\|^2 \right)$$

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$$\hat{\boldsymbol{\beta}}_1 = \left(\mathbb{X}_1'\mathbf{M}_2\mathbb{X}_1\right)^{-1}\left(\mathbb{X}_1'\mathbf{M}_2\mathbf{Y}\right)$$

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- When will $\widehat{\boldsymbol{\beta}}_1$ will be the same regardless of whether $\mathbb{X}_2$ is included?

Why does residual regression work?

- We can find $\widehat{\boldsymbol{\beta}}_1$ by nested minimization:

$$\widehat{\boldsymbol{\beta}}_1 = \arg\min_{\boldsymbol{\beta}_1} \left(\min_{\boldsymbol{\beta}_2} \|\mathbf{Y} - \mathbb{X}_1\boldsymbol{\beta}_1 - \mathbb{X}_2\boldsymbol{\beta}_2\|^2 \right)$$

  - First find the minimum of the SSR over $\boldsymbol{\beta}_2$ fixing $\boldsymbol{\beta}_1$
  - Then find $\boldsymbol{\beta}_1$ that minimizes the resulting SSR.

- The projection and annihilator matrices are defined only by covariates.
  - $\mathbf{M}_2 = \mathbf{I}_n - \mathbb{X}_2(\mathbb{X}_2'\mathbb{X}_2)^{-1}\mathbb{X}_2'$
  - Creates residuals from a regression on or $\mathbb{X}_2$

- Solving the nested minimization gives:

$$\widehat{\boldsymbol{\beta}}_1 = (\mathbb{X}_1'\mathbf{M}_2\mathbb{X}_1)^{-1} (\mathbb{X}_1'\mathbf{M}_2\mathbf{Y})$$

- When will $\widehat{\boldsymbol{\beta}}_1$ will be the same regardless of whether $\mathbb{X}_2$ is included?
  - If $\mathbb{X}_1$ and $\mathbb{X}_2$ are orthogonal so $\mathbb{X}_2'\mathbb{X}_1 = 0$ so $\mathbf{M}_2\mathbb{X}_1 = \mathbb{X}_1$

Residual regression

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  - Intuition: residuals are orthogonal
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  - Sample version of the results we saw for the linear projection.

5/ Influential observations

Outliers, leverage points, and influential observations

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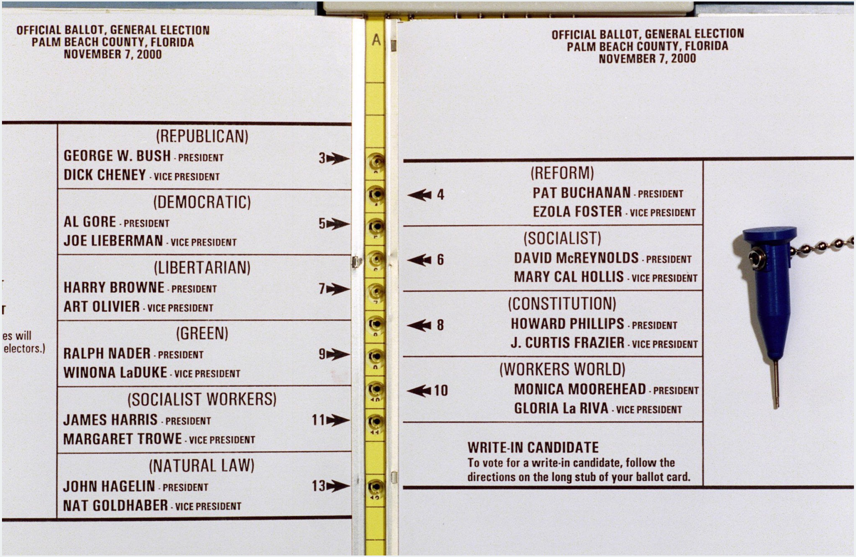
  3. **Influence point**: extreme in both directions

Example: Buchanan votes in Florida, 2000

- 2000 Presidential election in FL (Wand et al., 2001, APSR)



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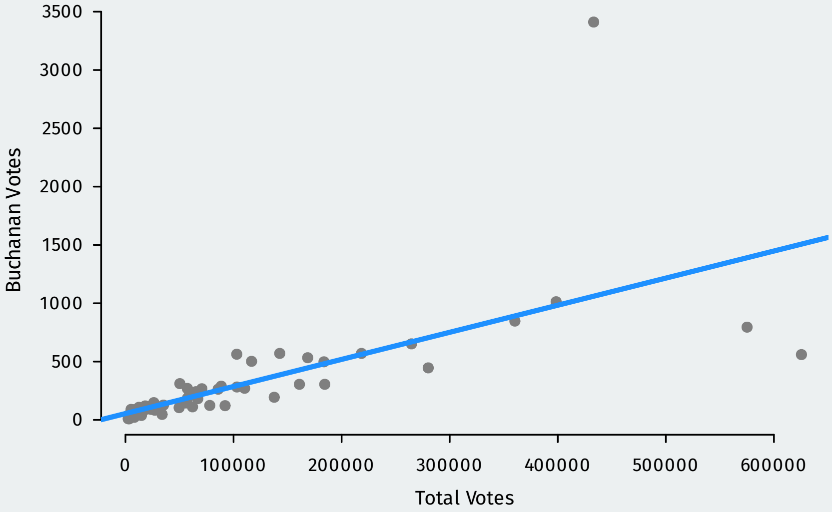

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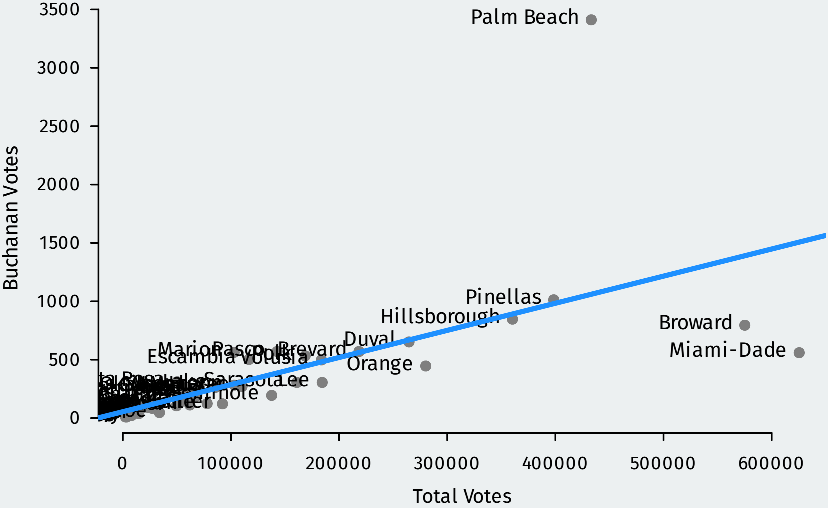
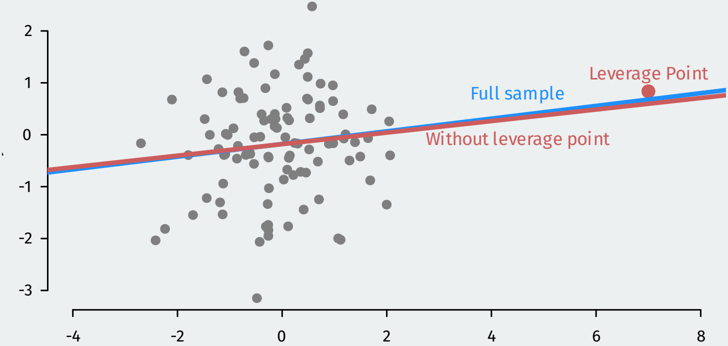


Example: Buchanan votes

```r
mod <- lm(edaybuchanan ~ edaytotal, data = flvote)
summary(mod)
```

```
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) 54.22945   49.14146    1.10     0.27
## edaytotal    0.00232    0.00031    7.48  2.4e-10 ***
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 333 on 65 degrees of freedom
## Multiple R-squared:  0.463,  Adjusted R-squared:  0.455
## F-statistic:   56 on 1 and 65 DF,  p-value: 2.42e-10
```

Leverage point definition



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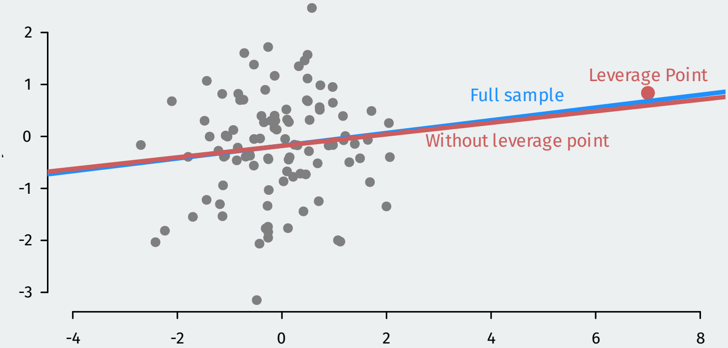

- Values that are extreme in the X dimension
- That is, values far from the center of the covariate distribution

Leverage values

- Let h_{ij} be the (i, j) entry of $\mathbf{P}$. Then:

$$\widehat{\mathbf{Y}} = \mathbf{PY} \qquad \Longrightarrow \qquad \widehat{Y}_i = \sum_{j=1}^{n} h_{ij} Y_j$$

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$$h_{ii} = \frac{1}{n} + \frac{(X_i - \overline{X})^2}{\sum_{j=1}^{n}(X_j - \overline{X})^2}$$

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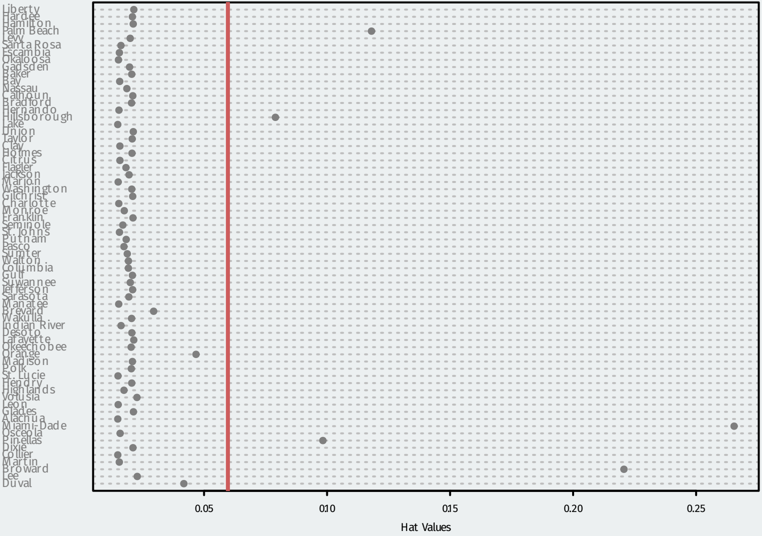
  - $\rightsquigarrow$ how far i is from the center of the X distribution
- **Rule of thumb:** examine hat values greater than $2(k+1)/n$

Buchanan hats

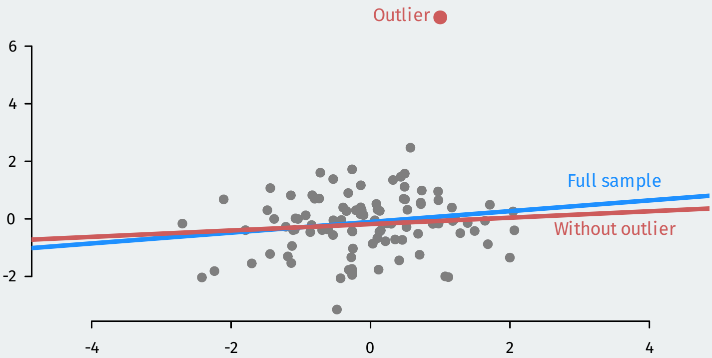
```r
head(hatvalues(mod), 5)
```

```
##      1      2      3      4      5
## 0.0418 0.0228 0.2207 0.0156 0.0149
```

Buchanan hats



Outlier definition



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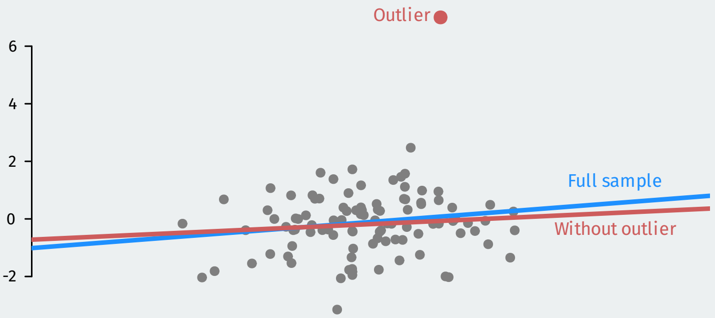

- An **outlier** is far away from the center of the Y distribution.
- Intuitively: a point that would be poorly predicted by the regression.

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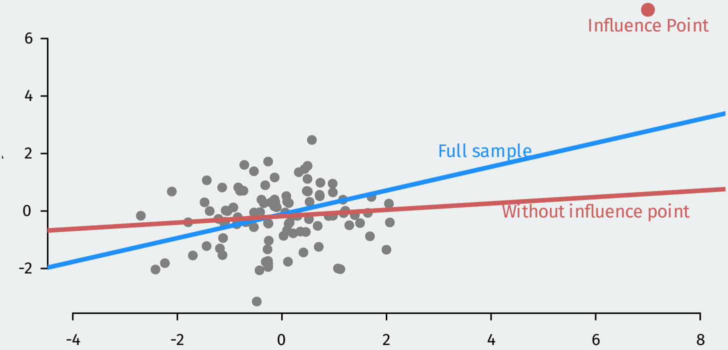
- Simple closed-form expressions:

$$\hat{\boldsymbol{\beta}}_{(-i)} = \hat{\boldsymbol{\beta}} - \left(\mathbb{X}'\mathbb{X}\right)^{-1}\mathbf{X}_i\tilde{e}_i \qquad \tilde{e}_i = \frac{\hat{e}_i}{1 - h_{ii}}$$

Influence points



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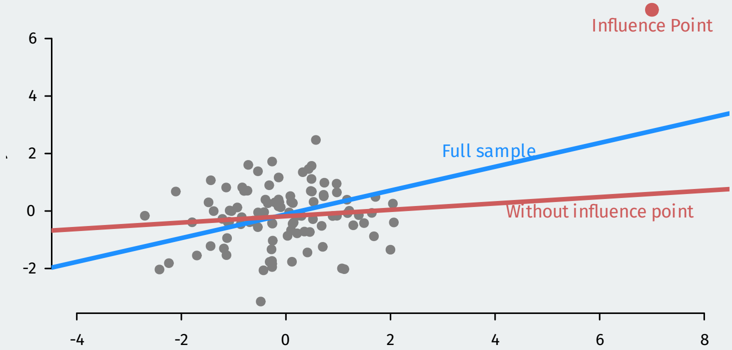

- An **influence point** is one that is both an outlier and a leverage point.
- Extreme in both the X and Y dimensions

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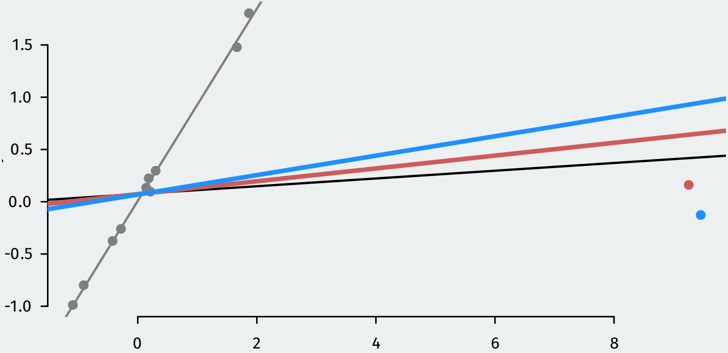
  - Does removing the point change a coefficient by a lot?

Limitations of the standard tools



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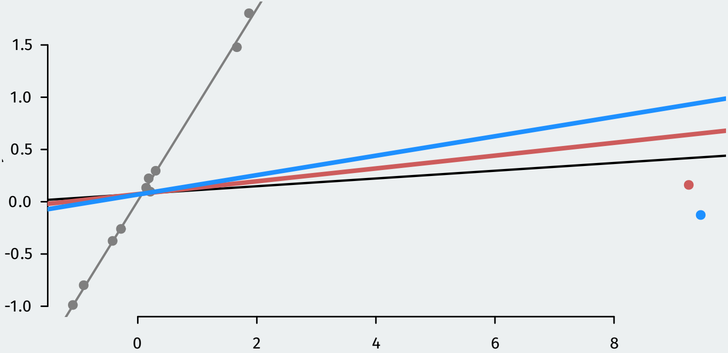

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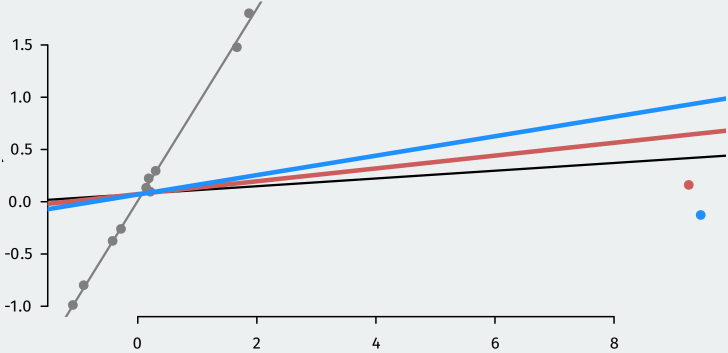

- What happens when there are two influence points?
- Red line drops the red influence point
- Blue line drops the blue influence point

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- Is the outlier part of the data generating process?
  - Transform the dependent variable ($\log(y)$)
  - Use a method that is robust to outliers (robust regression, least absolute deviations)