11. (Linear) Regression

Fall 2023

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Gov 2002 (Harvard)

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- Now: building to a specific estimator, least squares regression.
- First we need to understand what a “linear model” is and when/why we need it.
    - No estimators quite yet. First, let’s understand what we are estimating.
- Linear model is ubiquitous but poorly understood. Lots of subtlety here.

Regression derivatives and partial effects

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    - How do we decide what form $\mu(\mathbf{x})$ should take?

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- More generally for any discrete X_i:

$$\hat{\mu}(x) = \frac{\sum_{i=1}^{N} Y_i \mathbb{I}(X_i = x)}{\sum_{i=1}^{N} \mathbb{I}(X_i = x)}$$

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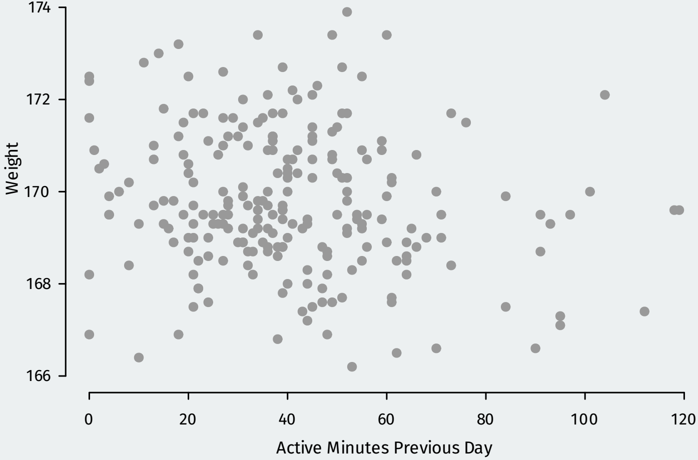
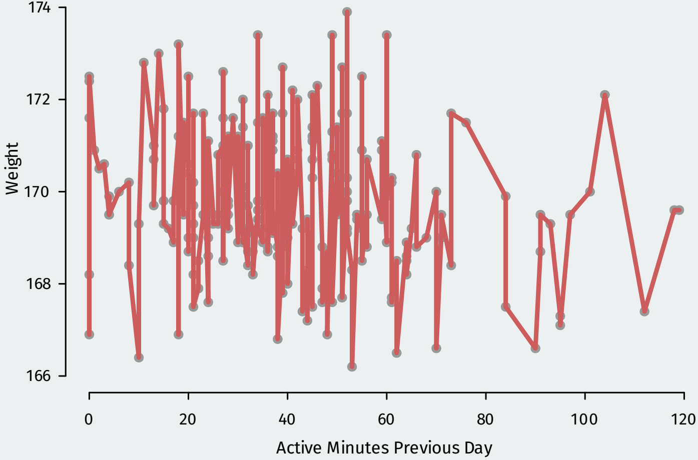
- Example:
  - Personal data science: I wear an activity tracker and have a smart scale.
  - Relationship between my weight and active minutes in the previous day.

Continuous covariate example



Continuous covariate CEF: interpolation



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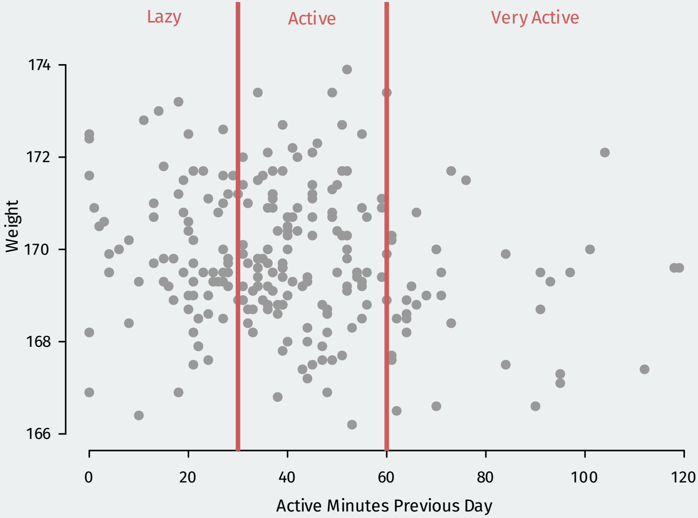

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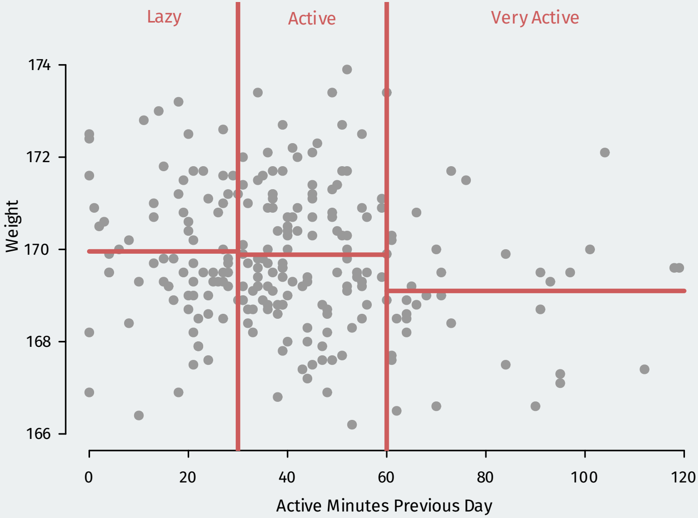

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- **Intercept**, β_0: the condition expectation of Y_i when $X_i = 0$
- **Slope**, β_1: change in the CEF of Y_i given a one-unit change in X_i

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- Effect of HS degree is the same as the effect of college degree.
- Put another way: average partial effects are constant $\frac{\partial \mu(x)}{\partial x} = \beta_1$

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  - Average partial effect of X_1 depends on X_2: $\partial\mu(x_1, x_2)/\partial x_1 = \beta_1 + x_2\beta_3$

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- >2 categories: dummies for all but category and everything is linear.

Linear CEF with multiple binary covariates

- What if we have two binary covariates, X_1 (race) and X_2 (1 urban/0 rural):

$$\mu(x_1, x_2) = \begin{cases} \mu_{00} & \text{if } x_1 = 0 \text{ and } x_2 = 0 \text{ (POC, rural)} \\ \mu_{10} & \text{if } x_1 = 1 \text{ and } x_2 = 0 \text{ (white, rural)} \\ \mu_{01} & \text{if } x_1 = 0 \text{ and } x_2 = 1 \text{ (POC, urban)} \\ \mu_{11} & \text{if } x_1 = 1 \text{ and } x_2 = 1 \text{ (white, urban)} \end{cases}$$

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    - $\beta_2 = \mu_{01} - \mu_{00}$: diff. in means for urban POC vs rural POC.

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- Interpretations:
    - $\beta_0 = \mu_{00}$: average wait times for rural POC.
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Linear CEF with multiple binary covariates

- What if we have two binary covariates, X_1 (race) and X_2 (1 urban/0 rural):

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- Generalizes to p binary variables if **all interactions included (saturated)**

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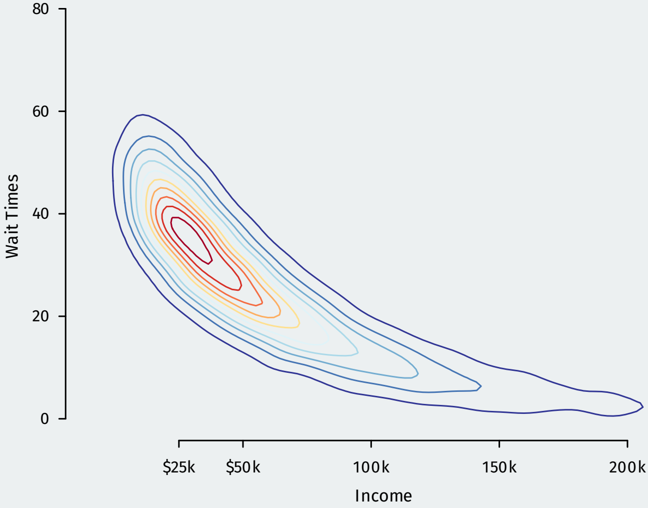

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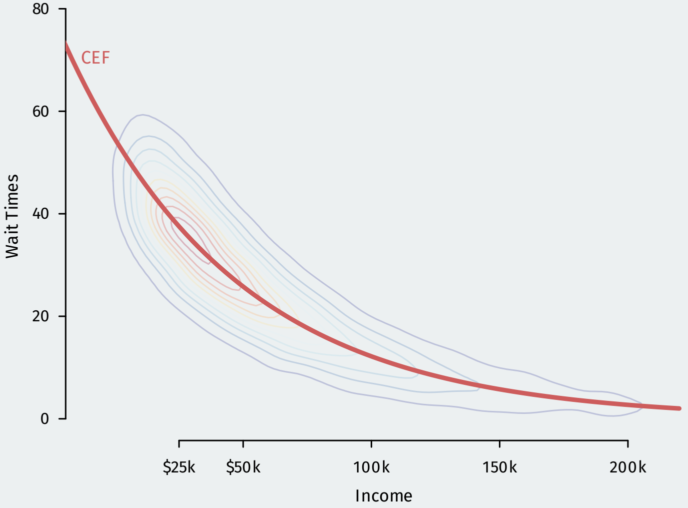

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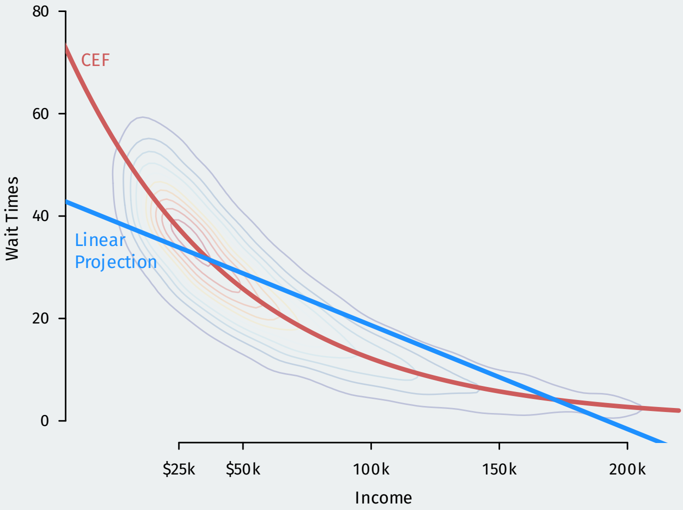

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- How to find $\boldsymbol{\beta}$? Minimize squared prediction error!

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- Holds for all values of x_2 and even if we add more variables.

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$$m(x_1, x_1^2, x_2) = \beta_0 + \beta_1 x_1 + \beta_2 x_1^2 + \beta_3 x_2,$$

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Partitioned Regression

$$(\alpha, \beta, \gamma) = \underset{(a,b,c)\in\mathbb{R}^3}{\arg\min}\ \mathbb{E}[(Y_i - (a + bX_i + cZ_i))^2]$$

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  - The relationship captured by β is between the outcome and the variation in X_i not linearly explained by Z_i

Partition regression more generally

- More general linear projection coefficients:

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Omitted variable bias

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