9. Asymptotics

Fall 2023

Matthew Blackwell

Gov 2002 (Harvard)

Where are we? Where are we going?

- Last time: introducing estimators, looking at finite-sample properties.

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- Last time: introducing estimators, looking at finite-sample properties.
- Now: can we say more as sample size grows?

Political canvassing study

POLITICAL SCIENCE

Durably reducing transphobia: A field experiment on door-to-door canvassing

David Broockman[1*] and Joshua Kalla[2]

Existing research depicts intergroup prejudices as deeply ingrained, requiring intense intervention to lastingly reduce. Here, we show that a single approximately 10-minute conversation encouraging actively taking the perspective of others can markedly reduce prejudice for at least 3 months. We illustrate this potential with a door-to-door canvassing intervention in South Florida targeting antitransgender prejudice. Despite declines in homophobia, transphobia remains pervasive. For the intervention, 56 canvassers went door to door encouraging active perspective-taking with 501 voters at voters' doorsteps. A randomized trial found that these conversations substantially reduced transphobia, with decreases greater than Americans' average decrease in homophobia from 1998 to 2012. These effects persisted for 3 months, and both transgender and nontransgender canvassers were effective. The intervention also increased support for a nondiscrimination law, even after exposing voters to counterarguments.

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Translating into math

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- Today: what happens to the distribution of $\widehat{\tau}_n$ as n grows?

1/ Asymptotics

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    - What if the data isn't normal? What is the sampling distribution of $\overline{X}_n$?
- **Asymptotics**: approximate the sampling distribution of $\overline{X}_n$ as n gets big.

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- Note: this is a sequence of random variables!

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A sequence $\{a_n : n = 1, 2, ...\}$ has the **limit** a written $a_n \to a$ as $n \to \infty$ if for all $\delta > 0$ there is some $n_\delta < \infty$ such that for all $n \geq n_\delta$, $|a_n - a| \leq \delta$.

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- $\{a_n : n = 1, 2, ...\}$ is **bounded** if there is $b < \infty$ such that $|a_n| < b$ for all n.

Limit example: (n-1)/n

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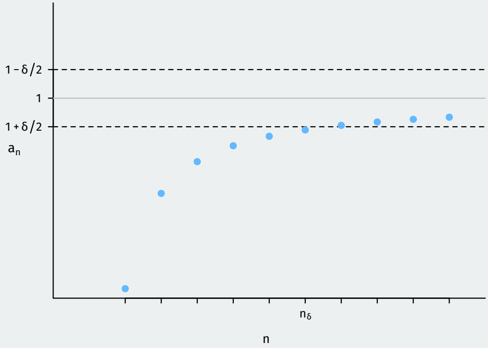

Convergence in Probability

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A sequence of random variables, $\{Z_n : n = 1, 2, ...\}$, is said to **converge in probability** to a value b if for every $\varepsilon > 0$,

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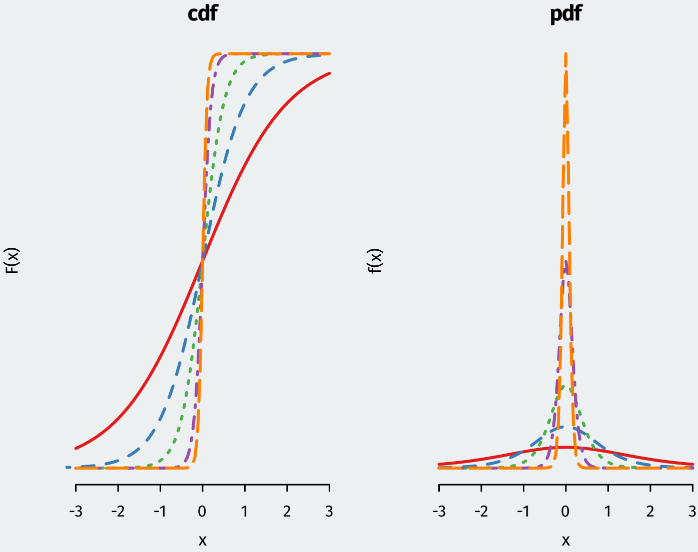
    - Inconsistent estimator are bad bad bad: more data gives worse answers!

Convergence in probability visually



Law of large numbers

Weak Law of Large Numbers

Let $X_1, \ldots, X_n$ be a an i.i.d. draws from a distribution with mean $\mathbb{E}[|X_i|] < \infty$. Let $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Then, $\overline{X}_n \xrightarrow{p} \mathbb{E}[X_i]$.

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- Implies general consistency of **plug-in estimators**
  - If $\mathbb{E}[|g(X_i)|] < \infty$, then $\frac{1}{n}\sum_{i=1}^{n} g(X_i) \overset{p}{\to} \mathbb{E}[g(X_i)]$

LLN by simulation in R

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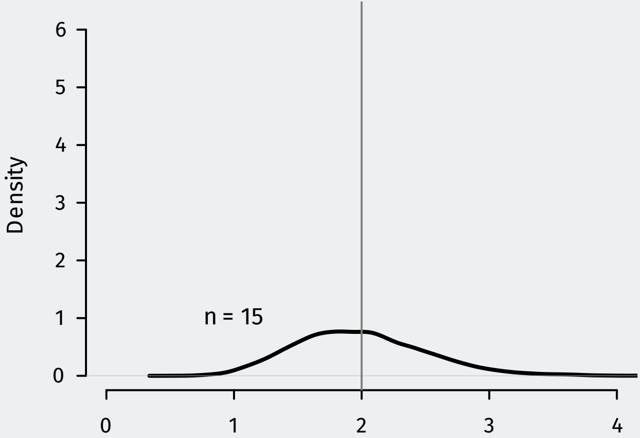
```
nsims <- 10000
holder <- matrix(NA, nrow = nsims, ncol = 6)
for (i in 1:nsims) {
  s5 <- rexp(n = 5, rate = 0.5)
  s15 <- rexp(n = 15, rate = 0.5)
  s30 <- rexp(n = 30, rate = 0.5)
  s100 <- rexp(n = 100, rate = 0.5)
  s1000 <- rexp(n = 1000, rate = 0.5)
  s10000 <- rexp(n = 10000, rate = 0.5)

  holder[i,1] <- mean(s5)
  holder[i,2] <- mean(s15)
  holder[i,3] <- mean(s30)
  holder[i,4] <- mean(s100)
  holder[i,5] <- mean(s1000)
  holder[i,6] <- mean(s10000)
}
```

LLN in action



- Distribution of $\overline{X}_{15}$

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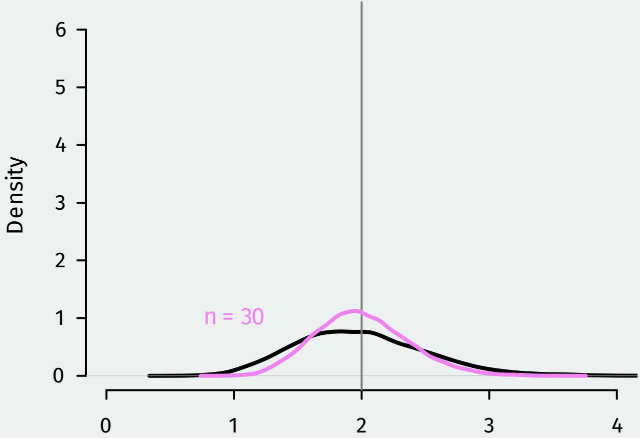

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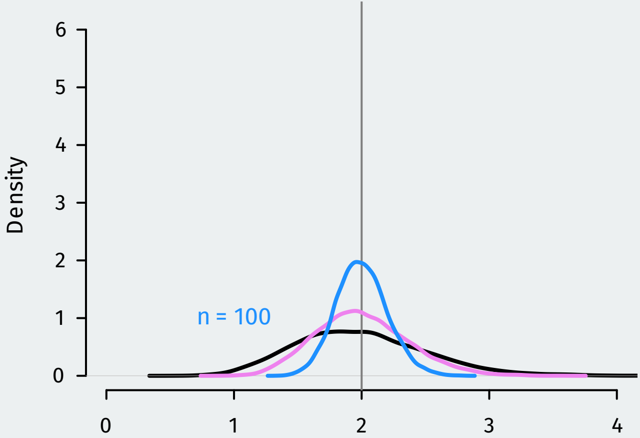

- Distribution of $\overline{X}_{100}$

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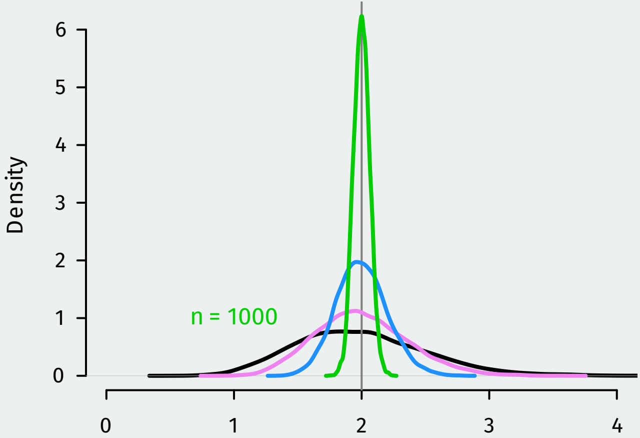

- Distribution of $\overline{X}_{1000}$

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- Variance places limits on how far an observation can be from its mean.

Proof of Chebyshev

- Let $Z = X - \mathbb{E}[X]$ with density $f_Z(x)$. Probability is just integral over the region:

$$\mathbb{P}\left(|Z| \geq \delta\right) = \int_{|x| \geq \delta} f_Z(x) dx$$

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- Under finite variance, applying this to $|\overline{X}_n - \mu|$ proves the LLN.

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- Interpretation: Under iid sampling, adding more units gets us closer and closer to the truth.

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2/ Central Limit Theorem

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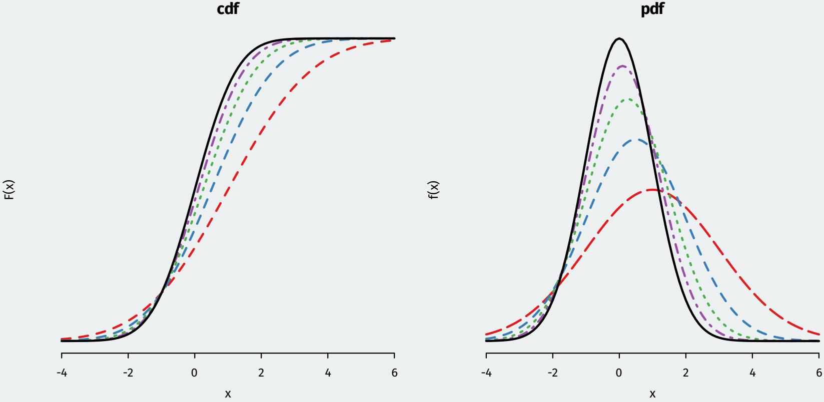
Convergence in distribution visualization

$$Z_n \sim N(1/n, 1 + 1/n) \xrightarrow{d} N(0,1)$$



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- But we can use the result to get an approximation: $\overline{X}_n \overset{a}{\sim} N(\mu, \sigma^2/n)$,

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  - $\sqrt{n}\left(\overline{X}_n - \mu\right)$ is more "stable" since its variance doesn't depend on n
- But we can use the result to get an approximation: $\overline{X}_n \overset{a}{\sim} N(\mu, \sigma^2/n)$,
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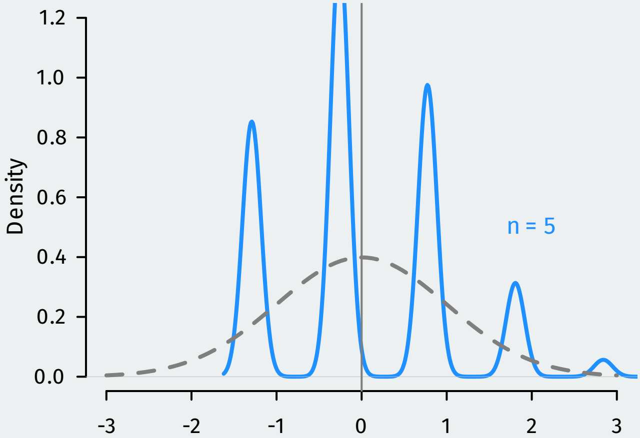
- $\rightsquigarrow$ approximations to probability statements about $\overline{X}_n$ when n is big!

CLT by simulation in R

```r
set.seed(02138)
nsims <- 10000
holder2 <- matrix(NA, nrow = nsims, ncol = 6)
for (i in 1:nsims) {
  s5 <- rbinom(n = 5, size = 1, prob = 0.25)
  s15 <- rbinom(n = 15, size = 1, prob = 0.25)
  s30 <- rbinom(n = 30, size = 1, prob = 0.25)
  s100 <- rbinom(n = 100, size = 1, prob = 0.25)
  s1000 <- rbinom(n = 1000, size = 1, prob = 0.25)
  s10000 <- rbinom(n = 10000, size = 1, prob = 0.25)

  holder2[i,1] <- mean(s5)
  holder2[i,2] <- mean(s15)
  holder2[i,3] <- mean(s30)
  holder2[i,4] <- mean(s100)
  holder2[i,5] <- mean(s1000)
  holder2[i,6] <- mean(s10000)
}
```

CLT in action



- Distribution of $\frac{\overline{X}_5 - \mu}{\sigma/\sqrt{5}}$

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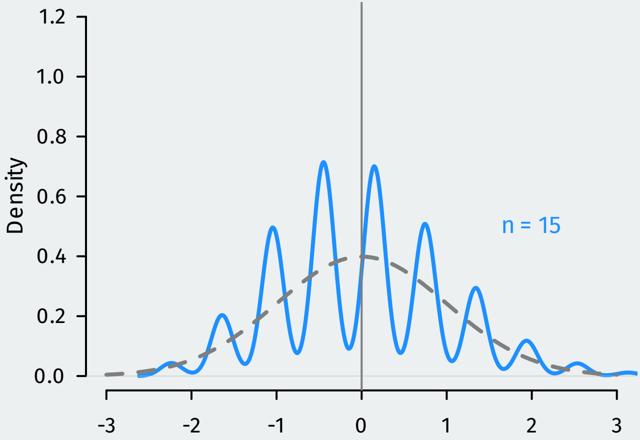

- Distribution of $\frac{\overline{X}_{15} - \mu}{\sigma / \sqrt{15}}$

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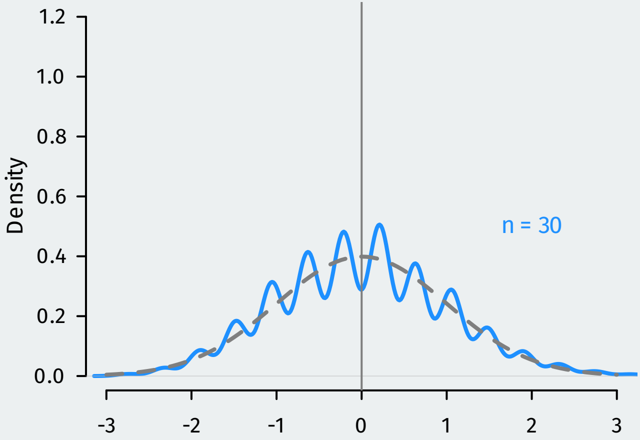

- Distribution of $\frac{\overline{X}_{30}-\mu}{\sigma/\sqrt{30}}$

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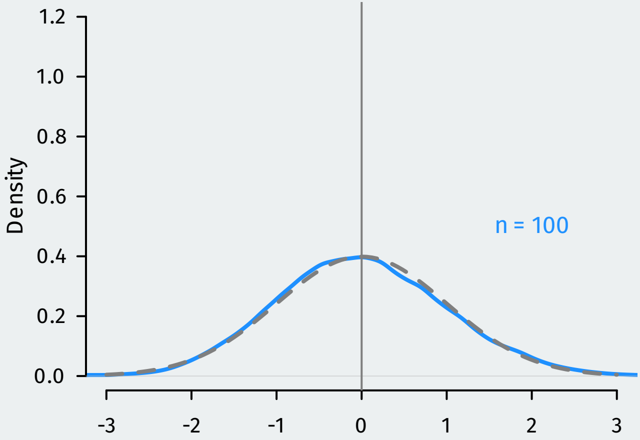

- Distribution of $\frac{\overline{X}_{100} - \mu}{\sigma / \sqrt{100}}$

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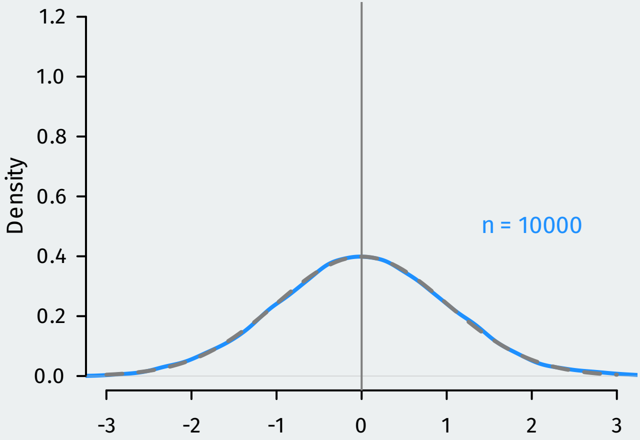

- Distribution of $\frac{\overline{X}_{10000}-\mu}{\sigma/\sqrt{10000}}$

CLT for plug-in estimators

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Why is asymptotic normality important?

- An estimator $\widehat{\theta}_n$ for θ is **asymptotically normal** when

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    - **Warning:** you do not know if you sample is big enough for this to be a good approximation.

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- Extremely useful when trying to figure out what the large-sample distribution of an estimator is.

Variance estimation with plug-in estimators

- Plug-in CLT:

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- We can show that $\widehat{V}_\theta \xrightarrow{p} V_\theta$ and so by Slutsky:

$$\frac{\sqrt{n}\left(\hat{\theta}_n - \theta\right)}{\sqrt{\widehat{V}_\theta}} \xrightarrow{d} \frac{\mathcal{N}(0, V_\theta)}{\sqrt{V_\theta}} \sim \mathcal{N}(0, 1)$$

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- Very common for when we're estimating multiple parameters $\boldsymbol{\theta}$ with $\hat{\boldsymbol{\theta}}_n$

3/ Confidence intervals

Interval estimation - what and why?

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- We can use the distribution of estimators (CLT!!) to derive these intervals.

What is a confidence interval?

Definition

A $1-\alpha$ **confidence interval** for a population parameter θ is a pair of statistics $L = L(X_1, \dots, X_n)$ and $U = U(X_1, \dots, X_n)$ such that $L < U$ and such that

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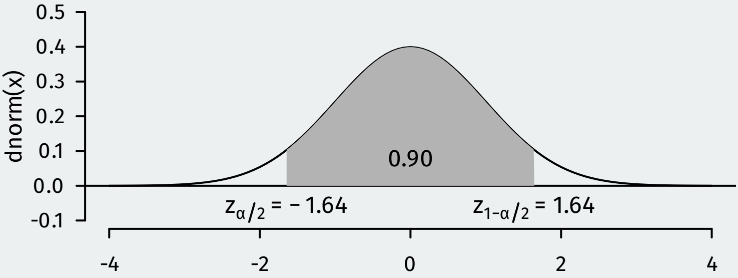
Finding the critical values



$$\mathbb{P}\left(-z_{1-\alpha/2} \leq \frac{\hat{\theta}_n - \theta}{\widehat{\text{se}}(\hat{\theta}_n)} \leq z_{1-\alpha/2}\right) \to 1-\alpha \quad \Longrightarrow \quad (1-\alpha) \text{ CI: } \hat{\theta}_n \pm z_{1-\alpha/2} \cdot \widehat{\text{se}}(\hat{\theta}_n)$$

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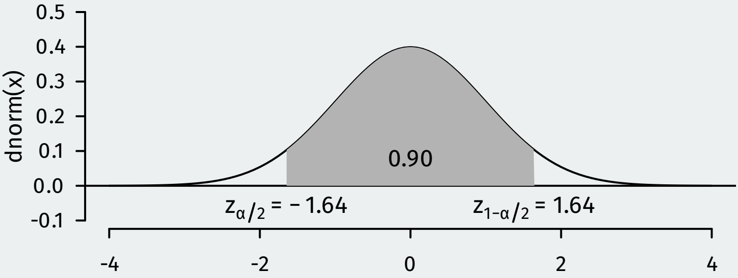

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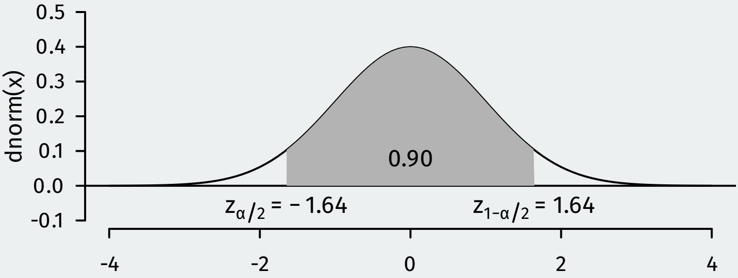

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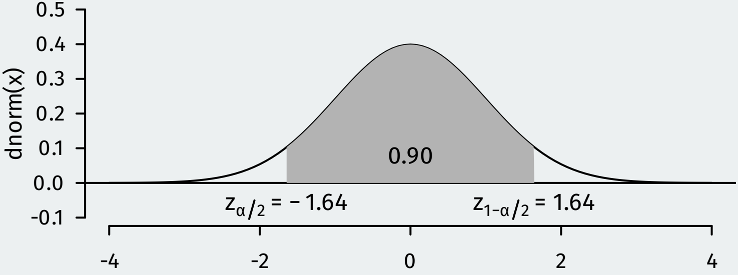

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    - Use the quantile function: $z_{1-\alpha/2} = \Phi^{-1}(1-\alpha/2)$ (`qnorm` in R)

CI for social pressure effect

TABLE 2. Effects of Four Mail Treatments on Voter Turnout in the August 2006 Primary Election

	Experimental Group				
	Control	Civic Duty	Hawthorne	Self	Neighbors
Percentage Voting	29.7%	31.5%	32.2%	34.5%	37.8%
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```r
neigh_var <- var(social$voted[social$treatment == "Neighbors"])
neigh_n <- 38201
civic_var <- var(social$voted[social$treatment == "Civic Duty"])
civic_n <- 38218

se_diff <- sqrt(neigh_var/neigh_n + civic_var/civic_n)

## c(lower, upper)
c((0.378 - 0.315) - 1.96 * se_diff, (0.378 - 0.315) + 1.96 * se_diff)
```

```
## [1] 0.0563 0.0697
```

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- Correct interpretation: **across 95% of random samples, the constructed confidence interval will contain the true value.**

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```
sims<- 10000
cover <- rep(0, times = sims)
low.bound <- up.bound <- rep(NA, times = sims)
for(i in 1:sims){
    draws <- rnorm(500, mean = 1, sd = sqrt(10))
    low.bound[i] <- mean(draws) - sd(draws) / sqrt(500) * 1.96
    up.bound[i] <- mean(draws) + sd(draws) / sqrt(500) * 1.96
    if (low.bound[i] < 1 & up.bound[i] > 1) {
        cover[i] <- 1
    }
}
mean(cover)
```

```
## [1] 0.95
```

Plotting the CIs

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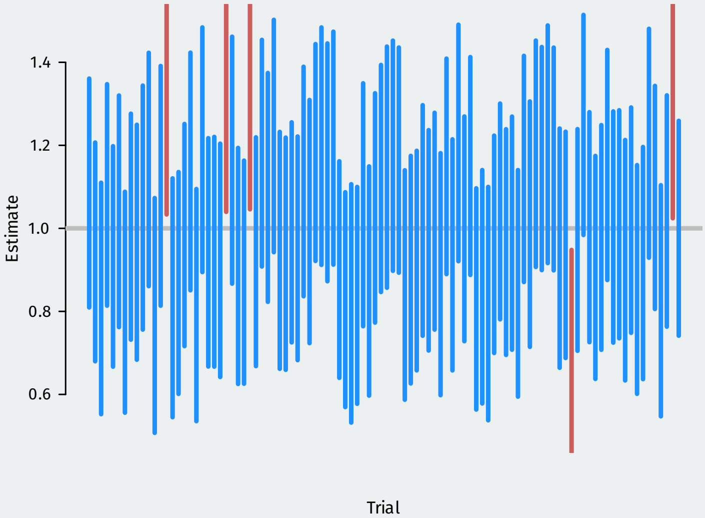

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4/ Delta method

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Bounded in probability

Definition

A random sequence Z_n is **bounded in probability**, written $Z_n = O_p(1)$ ("big-oh-p-one") for all $\delta > 0$ there exists a M_δ and n_δ, such that for $n \geq n_\delta$,

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- $Z_n = o_p(1)$ implies $Z_n = O_p(1)$ but not the reverse.

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Bounded in probability

Definition

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  - If $a_n^{-1} Z_n = O_p(1)$, we write $Z_n = O_p(a_n)$, so we have: $\hat{\theta}_n = \theta + O_p(n^{-1/2})$.