8. Sampling & Estimation

Fall 2023

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Gov 2002 (Harvard)

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- Last few weeks: probability, learning how to think about r.v.s
- Now: how to estimate features of underlying distributions with data.
- How do we construct estimators? What are their properties?

1/ Point Estimation

Motivating example

- Gerber, Green, and Larimer (APSR, 2008)

Dear Registered Voter:

WHAT IF YOUR NEIGHBORS KNEW WHETHER YOU VOTED?

Why do so many people fail to vote? We've been talking about the problem for years, but it only seems to get worse. This year, we're taking a new approach. We're sending this mailing to you and your neighbors to publicize who does and does not vote.

The chart shows the names of some of your neighbors, showing which have voted in the past. After the August 8 election, we intend to mail an updated chart. You and your neighbors will all know who voted and who did not.

DO YOUR CIVIC DUTY — VOTE!

MAPLE DR	Aug 04	Nov 04	Aug 06
9995 JOSEPH JAMES SMITH	Voted	Voted	______
9995 JENNIFER KAY SMITH		Voted	______
9997 RICHARD B JACKSON		Voted	______
9999 KATHY MARIE JACKSON		Voted	______

Motivating Example

```r
load("../assets/gerber_green_larimer.RData")
## turn turnout variable into a numeric
social$voted <- 1 * (social$voted == "Yes")
neigh.mean <- mean(social$voted[social$treatment == "Neighbors"])
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- Is this a “real”? Is it big?

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  - Which (if either) is better? How would we know?

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- Two metaphors:
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    - F represents the **data generating process**, we repeat n times
- **Statistical inference** or **learning** is using data to infer F.

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- **Point estimation**: providing a single "best guess" about these parameters.

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- An **estimate** is one particular realization of the estimator
  - Why is the following statement wrong: “My estimate was the sample mean and my estimator was 0.38”?

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  - $\hat{\theta}_n = 3$ always guess 3

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  - the 0.38 sample mean in the "Neighbors" group is one draw from this distribution

Sampling distribution, in pictures

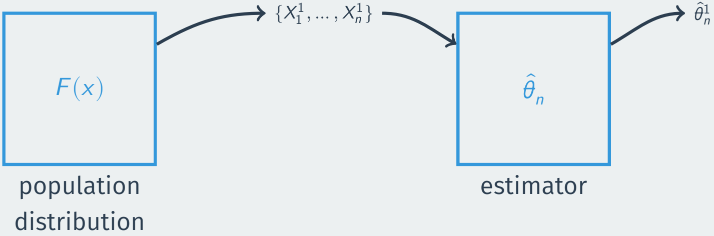
$\boxed{F(x)}$

population distribution

$\boxed{\hat{\theta}_n}$

estimator

Sampling distribution, in pictures



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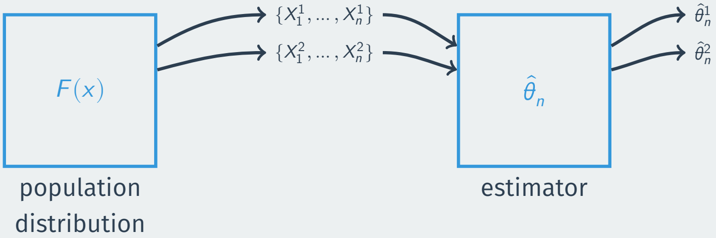

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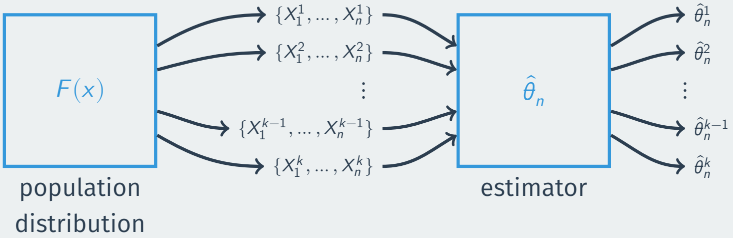

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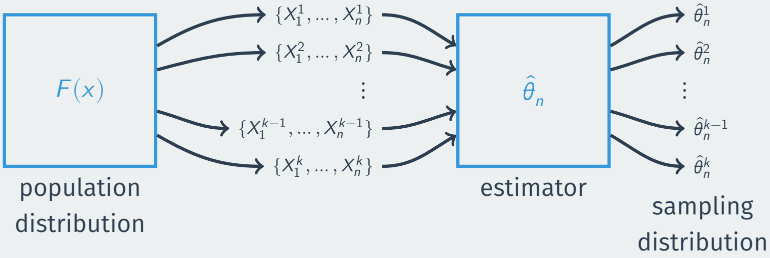

Sampling distribution

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## now we take the mean of one sample, which is
## one draw from the  **sampling distribution**
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```r
## Let's feed this sample to the sample mean estimator
## to get another estimate, which is another draw from
## the sampling distribution
mean(my.samp.2)
```

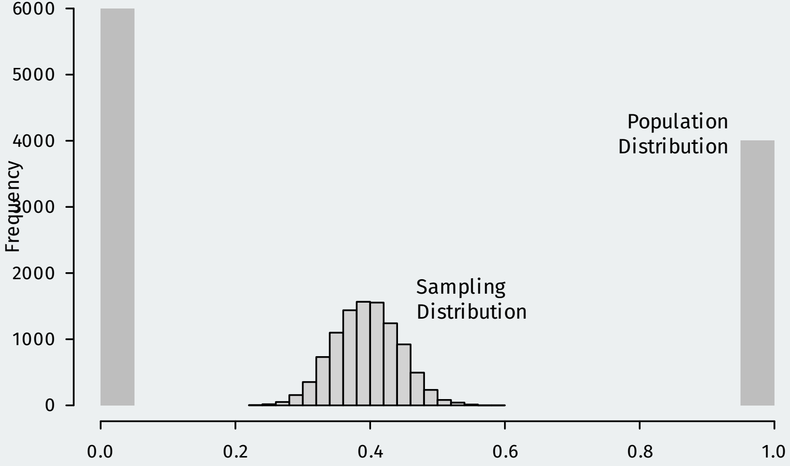
```
## [1] 0.5
```

Sampling distribution by simulation

- Let's generate 10,000 draws from the sampling distribution of the sample mean here when $n = 100$.

```r
nsims <- 10000
mean.holder <- rep(NA, times = nsims)
for (i in 1:nsims) {
    my.samp <- rbinom(n = 100, size = 1, prob = 0.4)
    mean.holder[i] <- mean(my.samp) ## sample mean
    first.holder[i] <- my.samp[1] ## first obs
}
```

Sampling distribution versus population distribution



Question The sampling distribution refers to the distribution of θ, true or false.

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  - $\rightsquigarrow$ if $\theta = \mathbb{E}[g(X)]$ replace $\mathbb{E}$ sample means: $\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} g(X_i)$

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- Covariance:

$$\sigma_{xy} = \mathrm{Cov}[X_i, Y_i] = \mathbb{E}[(X_i - \mathbb{E}[X_i])(Y_i - \mathbb{E}[Y_i])] \rightsquigarrow \widehat{\sigma}_{xy} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})(Y_i - \overline{Y})$$

2/ Finite-Sample Properties of Estimators

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    - **Large sample**: the properties of the sampling distribution as we let $n \to \infty$.

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- Might accept some bias for large reductions in variance for lower overall MSE.

3/ Design-based inference

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- Different **sampling designs** lead to different inclusion probabilities and difference inferences.

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- Remember: unbiased across repeated samples from the sampling design.

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- We can show that this is unbiased so that $\mathbb{E}[\widehat{\mathbb{V}}[\overline{X}_n]] = \mathbb{V}[\overline{X}_n]$

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