

7. Conditional Expectation

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Gov 2002 (Harvard)

Where are we? Where are we going?

- Covered most aspects of multivariate distributions.
- Time to preview a feature of these distributions we'll care a lot about: conditional expectations.
- At its core: how the average of one variable varies with others.

Defining condition expectations

Definition

The **conditional expectation** of Y conditional on $\mathbf{X} = \mathbf{x}$ is:

$$\mu(\mathbf{x}) = \mathbb{E}[Y \mid \mathbf{X} = \mathbf{x}] = \begin{cases} \sum_y y \mathbb{P}(Y = y \mid \mathbf{X} = \mathbf{x}) & \text{discrete } Y \\ \int_{-\infty}^{\infty} y f_{Y|\mathbf{X}}(y \mid \mathbf{x}) dy & \text{continuous } Y \end{cases}$$

- Expected value of the conditional distribution of Y given $\mathbf{X} = \mathbf{x}$.
 - $\mathbf{X} = (X_1, X_2, \dots, X_k)$ is a random vector ($k = 1$ just an r.v.)
- Viewed as a function of \mathbf{x} , it is the **conditional expectation function (CEF)**
 - How does the average value of Y change given different levels of \mathbf{X} ?

Conditional expectation example

| | Support Gay Marriage $Y = 1$ | Oppose Gay Marriage $Y = 0$ |
|----------------|------------------------------------|-----------------------------------|
| Female $X = 1$ | 0.32 | 0.19 |
| Male $X = 0$ | 0.29 | 0.20 |

- Conditional expectation of gay marriage support Y among men $X = 0$?

$$\begin{aligned}\mathbb{E}[Y \mid X = 0] &= \sum_y y \mathbb{P}(Y = y \mid X = 0) \\ &= 0 \times \mathbb{P}(Y = 0 \mid X = 0) + 1 \times \mathbb{P}(Y = 1 \mid X = 0) \\ &= 1 \times \frac{0.29}{0.29 + 0.20} = 0.59\end{aligned}$$

- If Y is binary, then $\mathbb{E}[Y \mid X = x] = \mathbb{P}(Y = 1 \mid X = x)$

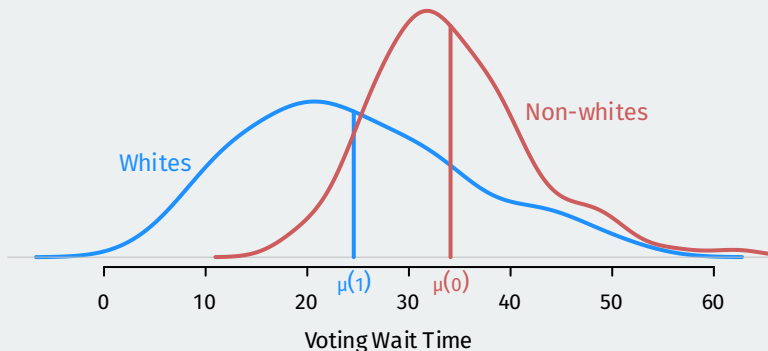
CEF for binary covariates

- Example:
 - Y_i is the time respondent i waited in line to vote.
 - $X_i = 1$ for whites, $X_i = 0$ for non-whites.
- Then the mean in each group is just a conditional expectation:

$$\mu(\text{white}) = E[Y_i | X_i = \text{white}]$$

$$\mu(\text{non-white}) = E[Y_i | X_i = \text{non-white}]$$

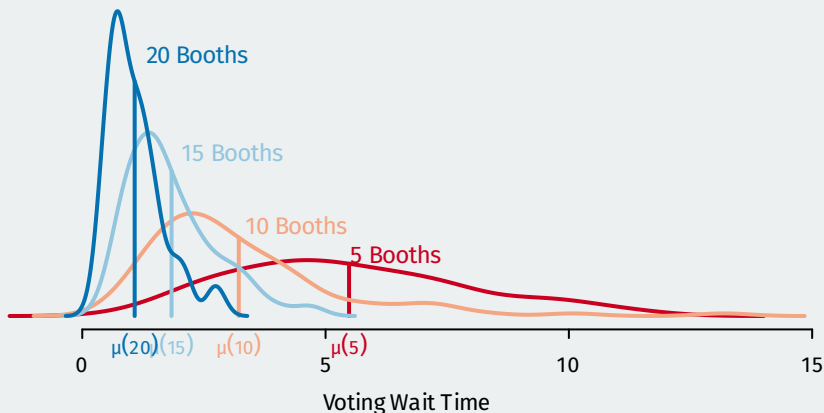
Why is the CEF useful?



- The CEF encodes relationships between variables.
- If $\mu(\text{white}) < \mu(\text{non-white})$, so that waiting times for whites are shorter on average than for non-whites.
- Indicates a relationship **in the population** between race and wait times.

CEF for discrete covariates

- New covariate: X_i is the # of polling booths at citizen i 's polling station.
- $\mu(x)$ is the mean of Y_i changes as X_i changes:



CEF with multiple covariates

- We can also CEF conditioning on multiple variables $\mu(\mathbf{x})$:

$$\mu(\text{white, man}) = \mathbb{E}[Y_i | X_i = \text{white}, Z_i = \text{man}]$$

$$\mu(\text{white, woman}) = \mathbb{E}[Y_i | X_i = \text{white}, Z_i = \text{woman}]$$

$$\mu(\text{non-white, man}) = \mathbb{E}[Y_i | X_i = \text{non-white}, Z_i = \text{man}]$$

$$\mu(\text{non-white, woman}) = \mathbb{E}[Y_i | X_i = \text{non-white}, Z_i = \text{woman}]$$

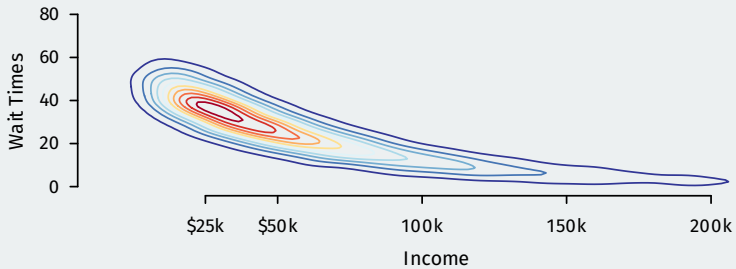
- Why? Allows more credible **all else equal** comparisons (ceteris paribus).
- Ex: average difference in wait times between white and non-white citizens **of the same gender**:

$$\mu(\text{white, man}) - \mu(\text{non-white, man})$$

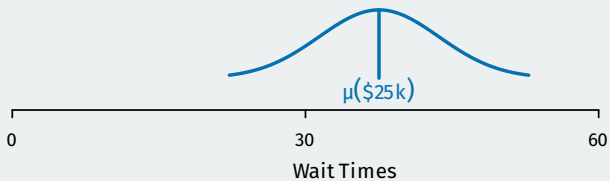
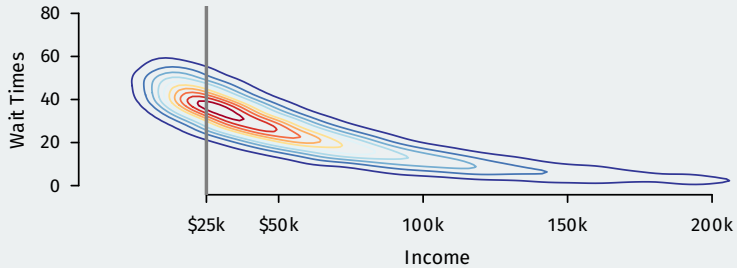
CEF for continuous covariates

- What if our independent variable, X_i is income?
- Many possible values of $X_i \rightsquigarrow$ many possible values of $\mathbb{E}[Y_i|X_i = x]$.
 - Writing out each value of the CEF no longer feasible.
- Now we will think about $\mu(x) = \mathbb{E}[Y_i|X_i = x]$ as function. What does this function look like:
 - Linear: $\mu(x) = \alpha + \beta x$
 - Quadratic: $\mu(x) = \alpha + \beta x + \gamma x^2$
 - Crazy, nonlinear: $\mu(x) = \alpha/(\beta + x)$
- These are **unknown functions in the population!** This is going to make producing an estimator $\hat{\mu}(x)$ very difficult!

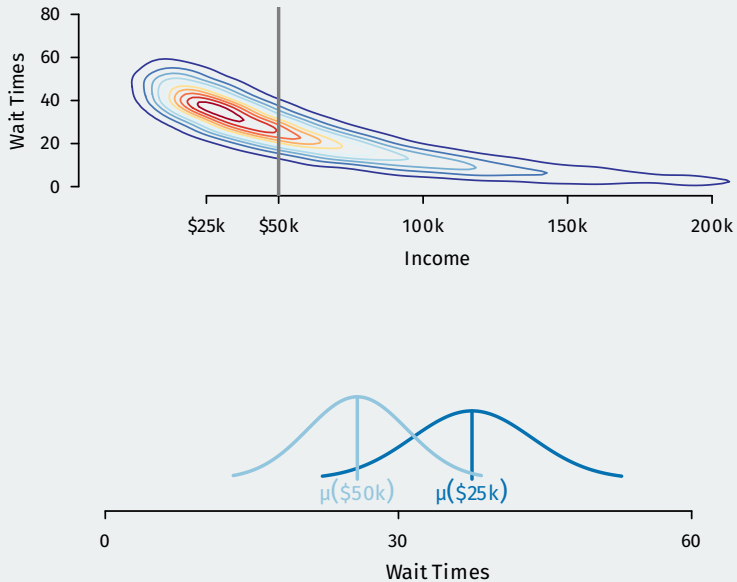
Wait times and income



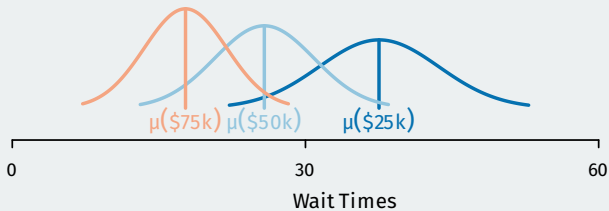
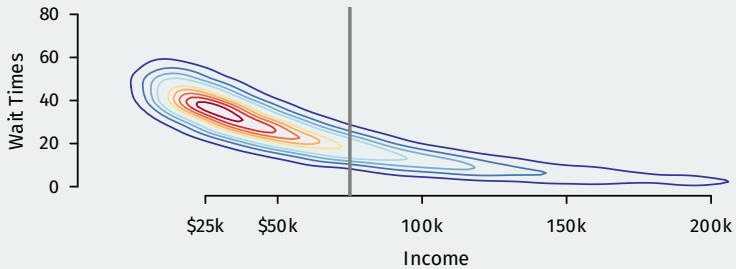
Wait times and income



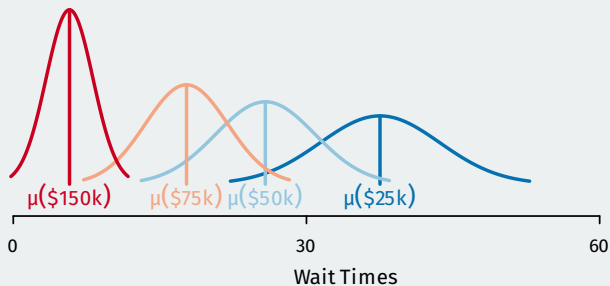
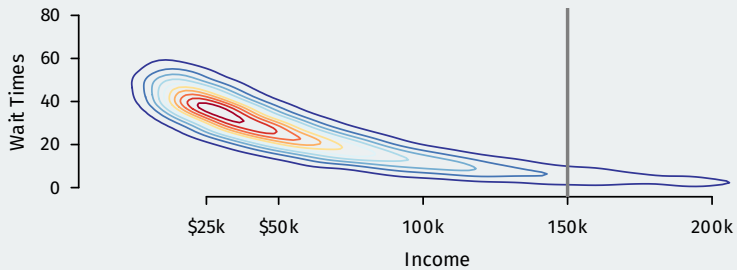
Wait times and income



Wait times and income



Wait times and income



Conditional expectations as random variables

- The conditional expectation is a function of \mathbf{x} : $\mu(\mathbf{x}) = \mathbb{E}[Y \mid \mathbf{X} = \mathbf{x}]$.
 - Not random: for a particular \mathbf{x} , $\mu(\mathbf{x})$ is a number.
 - Conditional expectation given an event.
- What about the conditional expectation given an r.v., $\mathbb{E}[Y \mid \mathbf{X}]$?
 - Why? Best prediction about Y given we get to know \mathbf{X} .
- Obtained by plugging r.v. into the CEF: $\mathbb{E}[Y \mid X] = \mu(X)$
- This is itself a random variable! For binary X :

$$\mathbb{E}[Y \mid X] = \begin{cases} \mu(0) & \text{with prob. } \mathbb{P}(X = 0) \\ \mu(1) & \text{with prob. } \mathbb{P}(X = 1) \end{cases}$$

- Has an expectation, $\mathbb{E}[\mathbb{E}[Y \mid X]]$, and a variance, $\mathbb{V}[\mathbb{E}[Y \mid X]]$.

Law of iterated expectations

Simple Law of Iterated Expectations

If $\mathbb{E}|Y| < \infty$, for any random vector \mathbf{X} , $\mathbb{E}\{\mathbb{E}[Y | \mathbf{X}]\} = E[Y]$.

- Expectation of the conditional expectation is the marginal expectation.
 - Discrete version: $\mathbb{E}[\mathbb{E}[Y | X]] = \sum_x \mathbb{E}[Y | X = x] \mathbb{P}(X = x) = \mathbb{E}[Y]$
 - Continuous version: $\mathbb{E}[\mathbb{E}[Y | X]] = \int_x \mathbb{E}[Y | X = x] f_X(x) dx = \mathbb{E}[Y]$
- General version allows for two conditioning sets:

Law of Iterated Expectations

If $\mathbb{E}|Y| < \infty$, for any random vectors \mathbf{X}_1 and \mathbf{X}_2 ,

$$\mathbb{E}\{\mathbb{E}[Y | \mathbf{X}_1, \mathbf{X}_2] | \mathbf{X}_1\} = E[Y | \mathbf{X}_1].$$

- “Averaging” over what is not constant (\mathbf{X}_2).

Example: law of iterated expectations

| | Support Gay Marriage $Y = 1$ | Oppose Gay Marriage $Y = 0$ | Marginal |
|----------------|------------------------------------|-----------------------------------|----------|
| Female $X = 1$ | 0.32 | 0.19 | 0.51 |
| Male $X = 0$ | 0.29 | 0.20 | 0.49 |
| Marginal | 0.61 | 0.39 | |

- $\mathbb{E}[Y | X = 1] = 0.62$ and $\mathbb{E}[Y | X = 0] = 0.59$.
- $\mathbb{P}(X = 1) = 0.51$ (females) and $\mathbb{P}(X = 0) = 0.49$ (males).
- Plug into the iterated expectations:

$$\begin{aligned}\mathbb{E}[\mathbb{E}[Y | X]] &= \mathbb{E}[Y | X = 0]\mathbb{P}(X = 0) + \mathbb{E}[Y | X = 1]\mathbb{P}(X = 1) \\ &= 0.59 \times 0.49 + 0.62 \times 0.51 = 0.605 = \mathbb{E}[Y]\end{aligned}$$

Properties of conditional expectations

1. $\mathbb{E}[c(X)Y | X] = c(X)\mathbb{E}[Y | X]$ for any function $c(X)$.

- Example: $\mathbb{E}[X^2Y | X] = X^2\mathbb{E}[Y | X]$ (If we know X , then we also know X^2)

2. If X and Y are independent r.v.s, then

$$\mathbb{E}[Y | X = x] = \mathbb{E}[Y].$$

3. If $X \perp\!\!\!\perp Y | Z$, then

$$\mathbb{E}[Y | X = x, Z = z] = \mathbb{E}[Y | Z = z]$$

4. Linearity: $\mathbb{E}[Y + X | Z] = \mathbb{E}[Y | Z] + \mathbb{E}[X | Z]$

CEF errors and projection

- CEF error: $e = Y - \mathbb{E}[Y | \mathbf{X}]$
- Properties of the CEF error:
 1. $\mathbb{E}[e | \mathbf{X}] = 0$
 2. $\mathbb{E}[e] = 0$
 3. If $\mathbb{E}[|Y|^r] < \infty$ for $r \geq 1$, then $\mathbb{E}[|e|^r] < \infty$
 4. For any function $h(\mathbf{X})$, $h(\mathbf{X})$ is uncorrelated with e : $\mathbb{E}[h(\mathbf{X})e] = 0$
- Last property: CEF errors are **orthogonal** to the space of functions of \mathbf{X} .
 - $\mathbb{E}[Y | \mathbf{X}]$ is the **projection** of Y on the space of all functions of \mathbf{X} .
 - Closest point in that space to Y .
- These properties are definitional, not assumptions.

Conditional Expectation as Best Predictor

- Suppose we want to predict Y based on random vector \mathbf{X} .
 - We can use any function $g(\mathbf{X})$ as our predictor.
- Mean squared error of our predictions:

$$\mathbb{E} \left[(Y - g(\mathbf{X}))^2 \right]$$

- What function will minimize this error? The CEF, $\mu(\mathbf{x})$!
- If $E[Y^2] < \infty$, then for any predictor $g(\mathbf{X})$,

$$\mathbb{E} \left[(Y - g(\mathbf{X}))^2 \right] \geq \mathbb{E} \left[(Y - \mu(\mathbf{X}))^2 \right]$$

Conditional Variance

Definition

The **conditional variance** of a Y given $\mathbf{X} = \mathbf{x}$ is defined as:

$$\sigma^2(\mathbf{x}) = \mathbb{V}[Y \mid \mathbf{X} = \mathbf{x}] = \mathbb{E}[(Y - \mu(\mathbf{x}))^2 \mid \mathbf{X} = \mathbf{x}]$$

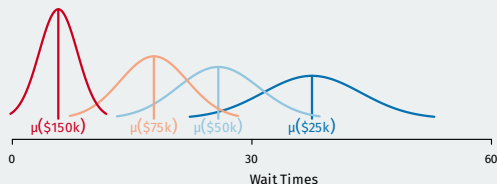
- Spread of the conditional distribution around its expectation.
- By definition, same as the variance of the CEF errors:

$$\mathbb{V}[Y \mid \mathbf{X} = \mathbf{x}] = \mathbb{V}[e \mid \mathbf{X} = \mathbf{x}] = \mathbb{E}[e^2 \mid \mathbf{X} = \mathbf{x}]$$

- Can re-express in the usual way:

$$\mathbb{V}[Y \mid \mathbf{X} = \mathbf{x}] = \mathbb{E}[Y^2 \mid \mathbf{X} = \mathbf{x}] - (\mathbb{E}[Y \mid \mathbf{X} = \mathbf{x}])^2$$

Skedasticity



- The error is **homoskedastic** if $\sigma^2(\mathbf{x}) = \sigma^2$ does not depend on \mathbf{x} .
 - Homoskedasticity greatly simplifies math, but often strong and implausible.
- The error is **heteroskedastic** if $\sigma^2(\mathbf{x})$ does depend on \mathbf{x}
 - Hetero = different, skedastic = scatter
- Default assumption should be the less restrictive one: heteroskedastic

Conditional variance as a random variable

- Conditional variance is just a function of \mathbf{x} : $\sigma^2(\mathbf{x}) = \mathbb{V}[Y | \mathbf{X} = \mathbf{x}]$
- $\sigma^2(\mathbf{X}) = \mathbb{V}[Y | \mathbf{X}]$ is an r.v. and a function of \mathbf{X} , just like $\mathbb{E}[Y | \mathbf{X}]$.
- With a binary X :

$$\mathbb{V}[Y | \mathbf{X}] = \begin{cases} \sigma^2(0) & \text{with prob. } \mathbb{P}(X = 0) \\ \sigma^2(1) & \text{with prob. } \mathbb{P}(X = 1) \end{cases}$$

- **Theorem** (Law of Total Variance/EVE's law):

$$\mathbb{V}[Y] = \mathbb{E}[\mathbb{V}[Y | \mathbf{X}]] + \mathbb{V}[\mathbb{E}[Y | \mathbf{X}]]$$

- The total variance can be decomposed into:
 1. the average of the within group variance ($\mathbb{E}[\mathbb{V}[Y | \mathbf{X}]]$) and
 2. how much the average varies between groups ($\mathbb{V}[\mathbb{E}[Y | \mathbf{X}]]$).