7. Conditional Expectation

Fall 2023

Matthew Blackwell

Gov 2002 (Harvard)

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- Time to preview a feature of these distributions we'll care a lot about: conditional expectations.
- At its core: how the average of one variable varies with others.

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The **conditional expectation** of Y conditional on $\mathbf{X} = \mathbf{x}$ is:

$$\mu(\mathbf{x}) = \mathbb{E}[Y \mid \mathbf{X} = \mathbf{x}] = \begin{cases} \sum_y y\, \mathbb{P}(Y = y \mid \mathbf{X} = \mathbf{x}) & \text{discrete } Y \\ \int_{-\infty}^{\infty} y\, f_{Y|\mathbf{X}}(y \mid \mathbf{x}) dy & \text{continuous } Y \end{cases}$$

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- Viewed as a function of $\mathbf{x}$, it is the **conditional expectation function (CEF)**
    - How does the average value of Y change given different levels of $\mathbf{X}$?

Conditional expectation example

	Support Gay Marriage $Y = 1$	Oppose Gay Marriage $Y = 0$
Female $X = 1$	0.32	0.19
Male $X = 0$	0.29	0.20

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- If Y is binary, then $\mathbb{E}[Y \mid X = x] = \mathbb{P}(Y = 1 \mid X = x)$

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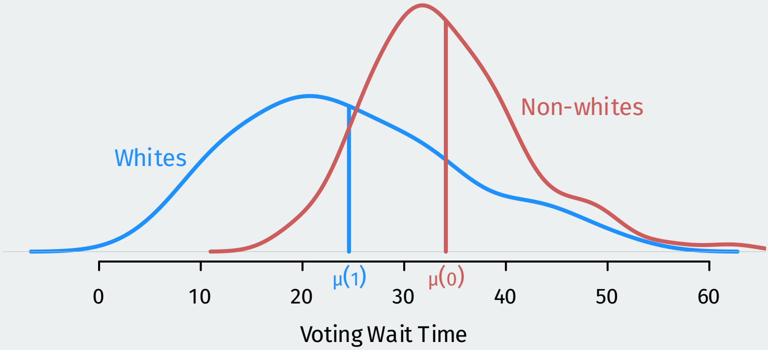
  - Y_i is the time respondent i waited in line to vote.
  - $X_i = 1$ for whites, $X_i = 0$ for non-whites.
- Then the mean in each group is just a conditional expectation:

$$\mu(\text{white}) = E[Y_i | X_i = \text{white}]$$
$$\mu(\text{non-white}) = E[Y_i | X_i = \text{non-white}]$$

Why is the CEF useful?



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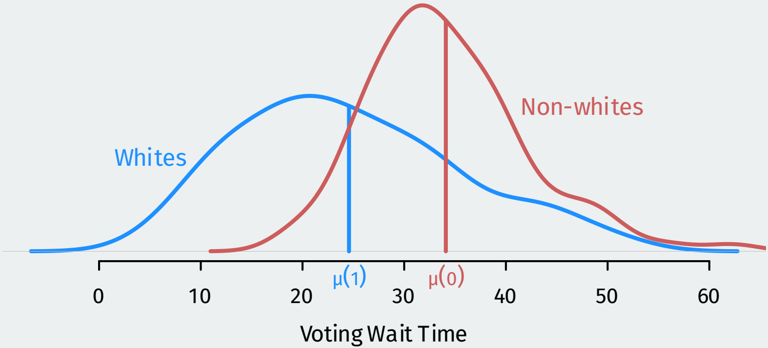

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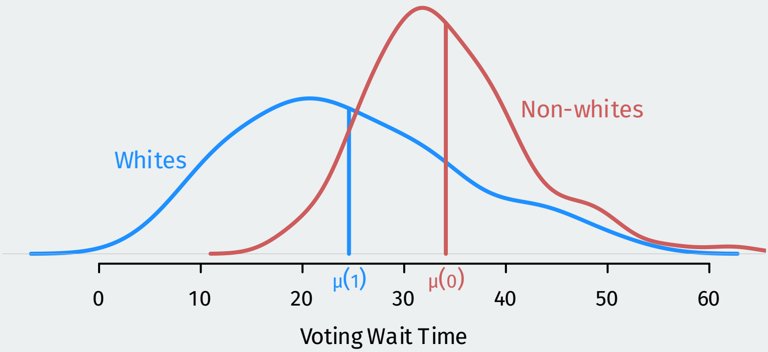

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- If $\mu(\text{white}) < \mu(\text{non-white})$, so that waiting times for whites are shorter on average than for non-whites.
- Indicates a relationship **in the population** between race and wait times.

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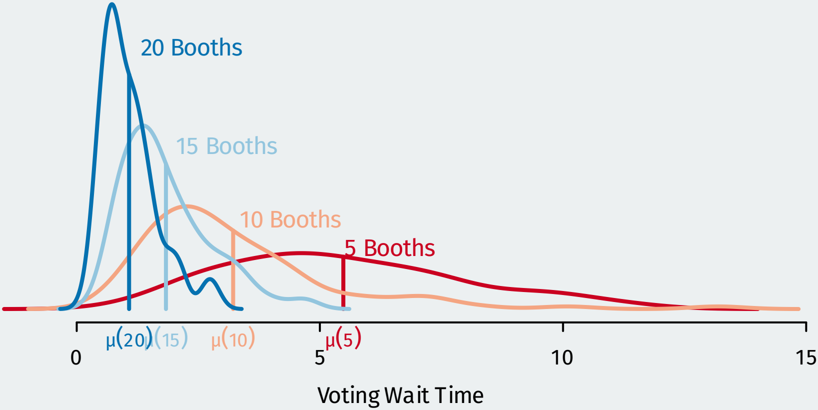

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- Ex: average difference in wait times between white and non-white citizens **of the same gender**:

$$\mu(\text{white}, \text{man}) - \mu(\text{non-white}, \text{man})$$

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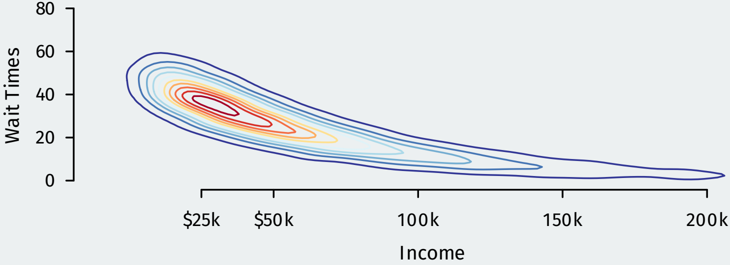
- These are **unknown functions in the population**! This is going to make producing an estimator $\hat{\mu}(x)$ very difficult!

Wait times and income



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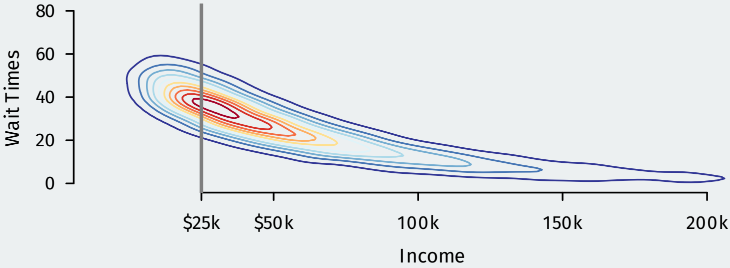

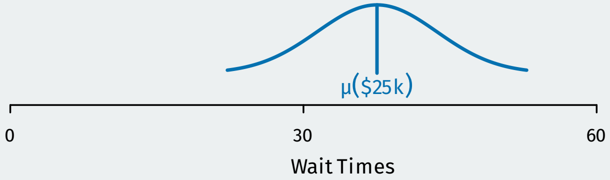

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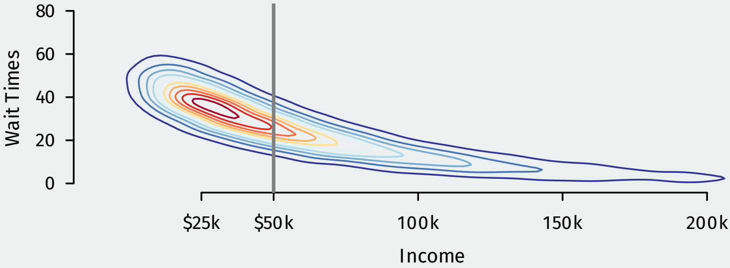

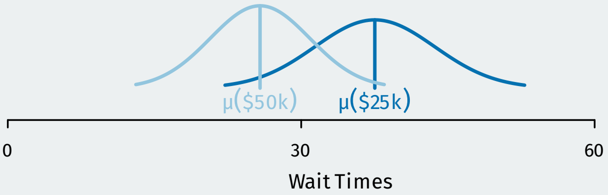

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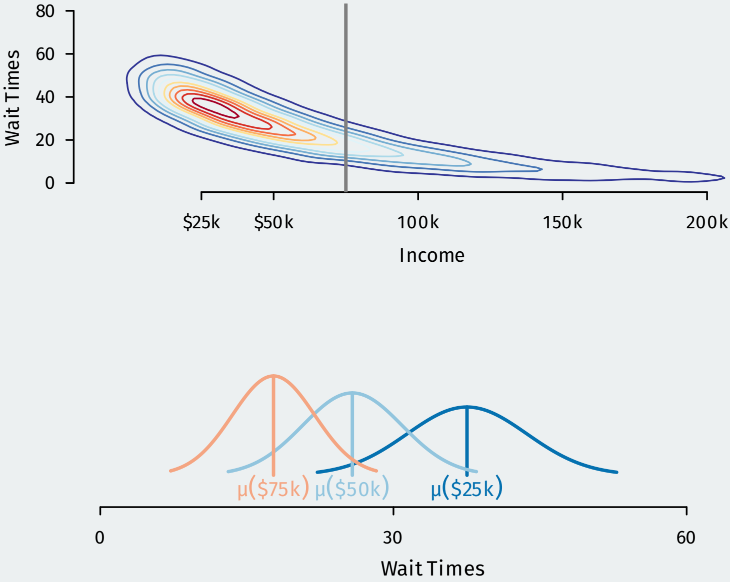

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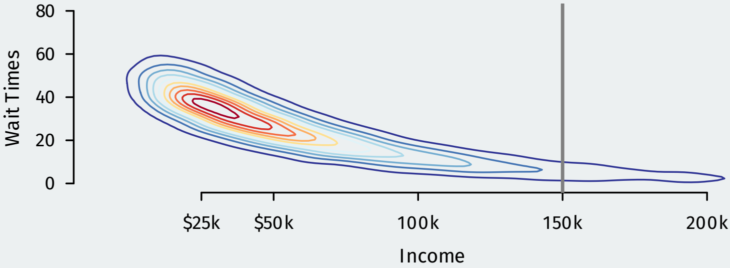

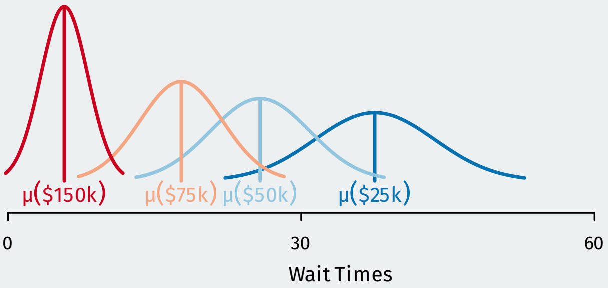

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- Has an expectation, $\mathbb{E}[\mathbb{E}[Y \mid X]]$, and a variance, $\mathbb{V}[\mathbb{E}[Y \mid X]]$.

Law of iterated expectations

Simple Law of Iterated Expectations

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$$\mathbb{E}\left\{\mathbb{E}[Y \mid \mathbf{X}_1, \mathbf{X}_2] \mid \mathbf{X}_1\right\} = E[Y \mid \mathbf{X}_1].$$

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  - Continuous version: $\mathbb{E}\left[\mathbb{E}[Y \mid X]\right] = \int_x \mathbb{E}[Y \mid X = x] f_X(x) dx = \mathbb{E}[Y]$
- General version allows for two conditioning sets:

Law of Iterated Expectations

If $\mathbb{E}|Y| < \infty$, for any random vectors $\mathbf{X}_1$ and $\mathbf{X}_2$,

$$\mathbb{E}\left\{\mathbb{E}[Y \mid \mathbf{X}_1, \mathbf{X}_2] \mid \mathbf{X}_1\right\} = E[Y \mid \mathbf{X}_1].$$

- “Averaging” over what is not constant ($\mathbf{X}_2$).

Example: law of iterated expectations

	Support Gay Marriage $Y = 1$	Oppose Gay Marriage $Y = 0$	Marginal
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- These properties are definitional, not assumptions.

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- If $E[Y^2] < \infty$, then for any predictor $g(\mathbf{X})$,

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Conditional Variance

Definition

The **conditional variance** of a Y given $\mathbf{X} =$ is defined as:

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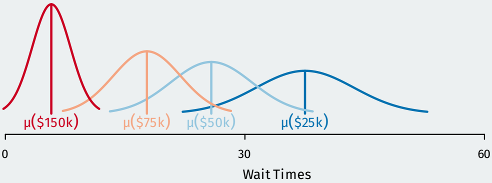
- Can re-express in the usual way:

$$\mathbb{V}[Y \mid \mathbf{X} = \mathbf{x}] = \mathbb{E}\left[Y^2 \mid \mathbf{X} = \mathbf{x}\right] - \left(\mathbb{E}[Y \mid \mathbf{X} = \mathbf{x}]\right)^2$$

Skedasticity



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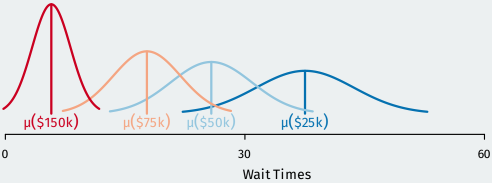

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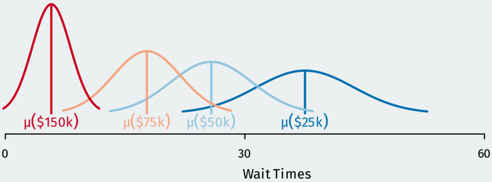

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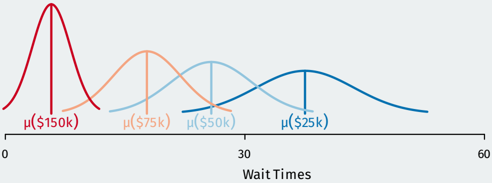

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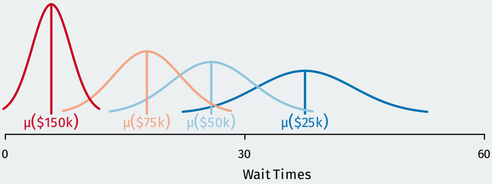

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- Default assumption should be the less restrictive one: heteroskedastic

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