6. Multivariate Distributions

Fall 2023

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Gov 2002 (Harvard)

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- Today: **distributions of multiple variables** to describe relationships between variables.
- Later: use data to **learn** about probability distributions.

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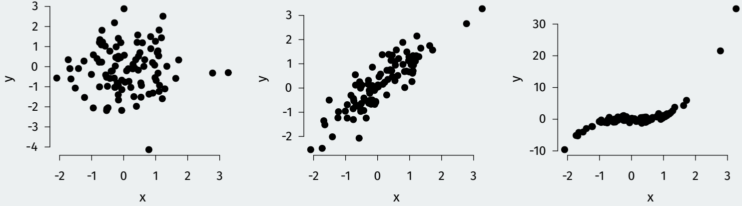
1. How to measure the relationship between two variables X and Y?
2. What if we have many observations of the same variable, $X_1, X_2, \dots, X_n$?

1/ Distributions of Multiple Random Variables

Joint distributions



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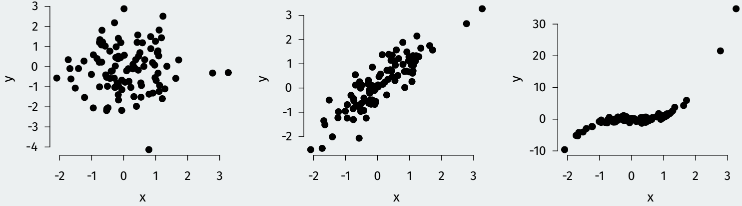
Joint distributions



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- Shape of the joint distribution $\rightsquigarrow$ the relationship between X and Y

Discrete r.v.s

Definition

The **joint probability mass function (p.m.f.)** of a pair of discrete r.v.s, (X, Y) describes the probability of any pair of values:

$$p_{X,Y}(x, y) = \mathbb{P}(X = x, Y = y)$$

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  - $\sum_x$ is shorthand for sum over all possible values of X

Example: Gay marriage and gender

	Support Gay Marriage $Y = 1$	Oppose Gay Marriage $Y = 0$
Female $X = 1$	0.32	0.19
Male $X = 0$	0.29	0.20

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$$p_{X,Y}(1, 1) = \mathbb{P}(X = 1, Y = 1) = 0.32$$

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    - Works because values of X are disjoint.

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Marginal	0.32 + 0.29		

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Marginal	0.61	0.19 + 0.20	

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The **conditional probability mass function** or conditional p.m.f. of Y conditional on X is

$$\mathbb{P}(Y = y \mid X = x) = \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(X = x)}$$

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- This is a valid univariate probability distribution!
  - $P(Y = y \mid X = x) \geq 0$ and $\sum_y \mathbb{P}(Y = y \mid X = x) = 1$
- Can define the **conditional expectation** of this p.m.f.:

$$E[Y \mid X = x] = \sum_y y\mathbb{P}(Y = y \mid X = x)$$

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$$\mathbb{P}(Y = 1 \mid X = 0)$$

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$$\mathbb{P}(Y = 1 \mid X = 0) = \frac{\mathbb{P}(X = 0, Y = 1)}{\mathbb{P}(X = 0)} = \frac{0.29}{0.29 + 0.20}$$

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$$\mathbb{P}(Y = 1 \mid X = 0) = \frac{\mathbb{P}(X = 0, Y = 1)}{\mathbb{P}(X = 0)} = \frac{0.29}{0.29 + 0.20} = 0.592$$

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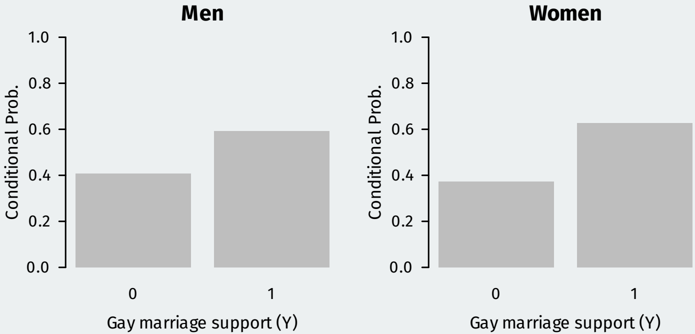

- Two values of $X \rightsquigarrow$ two **univariate** conditional distributions of Y

Bayes and LTP

- Bayes' rule for r.v.s:

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- Law of total probability for r.v.s:

$$\mathbb{P}(X = x) = \sum_{y} \mathbb{P}(X = x \mid Y = y)\mathbb{P}(Y = y)$$

Joint c.d.f.s

Definition

For two r.v.s X and Y, the **joint cumulative distribution function** or joint c.d.f. $F_{X,Y}(x, y)$ is a function such that for finite values x and y,

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- Well-defined for discrete and continuous X and Y.
- For discrete we simply have:

$$F_{X,Y}(x, y) = \sum_{i \leq x} \sum_{j \leq y} \mathbb{P}(X = i, Y = j)$$

Continuous r.v.s

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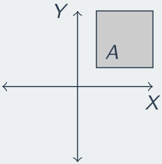
- Two continuous r.v.s: probability of being in some subset of the 2-dimensional plane.



Continuous joint p.d.f.

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If two continuous r.v.s X and Y with joint c.d.f. $F_{X,Y}$, their **joint p.d.f.** $f_{X,Y}(x,y)$ is the derivative of $F_{X,Y}$ with respect to x and y,

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- Integrate over both dimensions to get the probability of a region:

$$\mathbb{P}((X,Y) \in A) = \iint_{(x,y)\in A} f_{X,Y}(x,y) dx dy.$$

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- $\{(x, y) : f_{X,Y}(x, y) > 0\}$ is called the **support** of the distribution.

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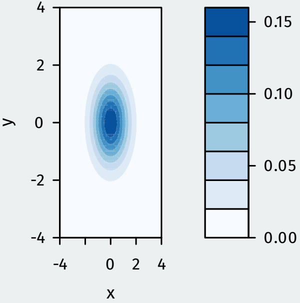
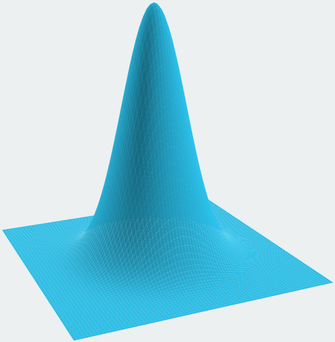
  2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx dy = 1$, (probabilities "sum" to 1)
- $\mathbb{P}(X = x, Y = y) = 0$ for similar reasons as with single r.v.s.

Joint densities are 3D





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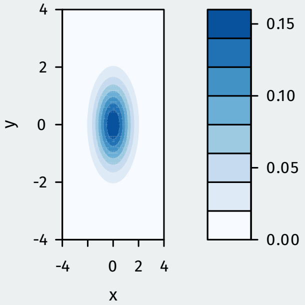
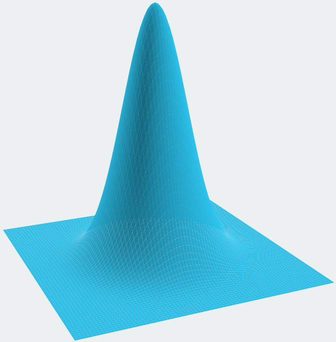
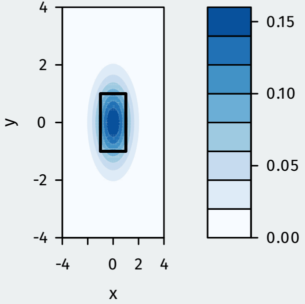
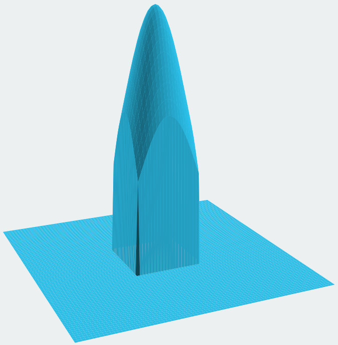




- X and Y axes are on the "floor," height is the value of $f_{X,Y}(x, y)$.
- Remember $f_{X,Y}(x, y) \neq \mathbb{P}(X = x, Y = y)$.

Probability = volume





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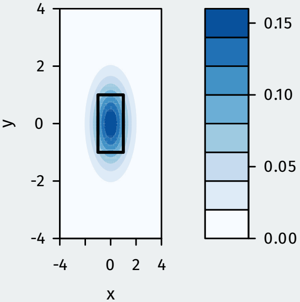

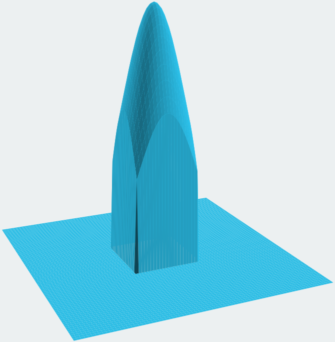

- $\mathbb{P}((X, Y) \in A) = \iint_{(x,y)\in A} f_{X,Y}(x, y)dxdy$
- Probability = volume above a specific region.

Continuous marginal distributions

- We can recover the marginal PDF of one of the variables by integrating over the distribution of the other variable:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx$$

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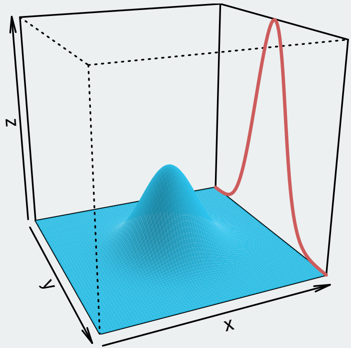
- Works for either variable:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy$$

Visualizing continuous marginals



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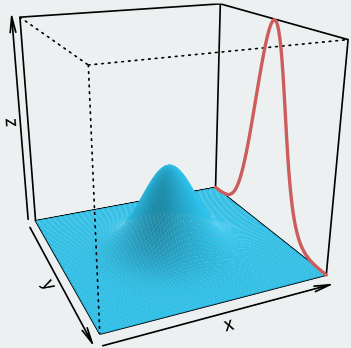

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- Pile up/flatten all of the joint density onto a single dimension.

Continuous conditional distributions

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for all values x s.t. $f_X(x) > 0$.

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Continuous conditional distributions

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- Based on the definition of the conditional p.m.f./p.d.f., we have the following factorization:

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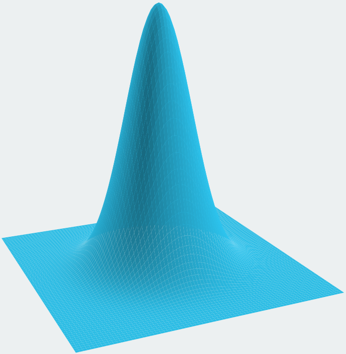
Conditional distributions as slices



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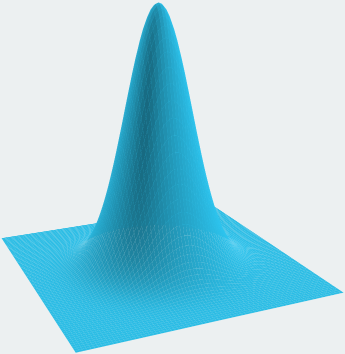

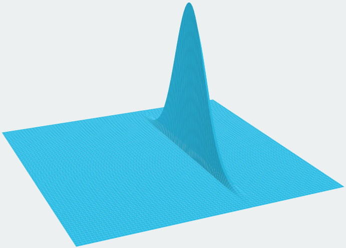

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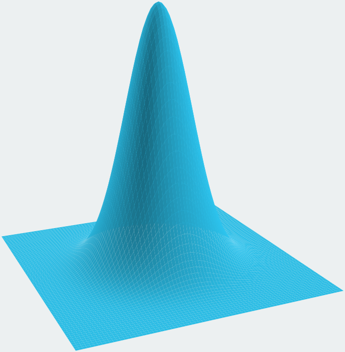

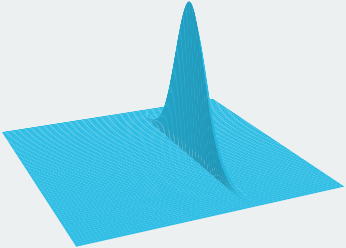

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- $f_{Y|X}(y|x_0)$ is proportional to joint p.d.f. along x_0: $f_{X,Y}(y, x_0)$
- Normalize by dividing by $f_X(x_0)$ to ensure proper p.d.f.

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- **Conditional independence** implies similar to conditional distributions:

$$\mathbb{P}(X \in A, Y \in B \mid Z) = \mathbb{P}(X \in A \mid Z)\mathbb{P}(Y \in B \mid Z)$$

2/ Expectations of Joint Distributions

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3/ Covariance and Correlation

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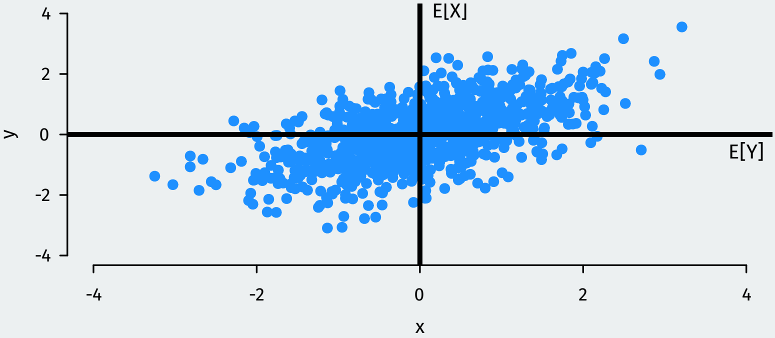

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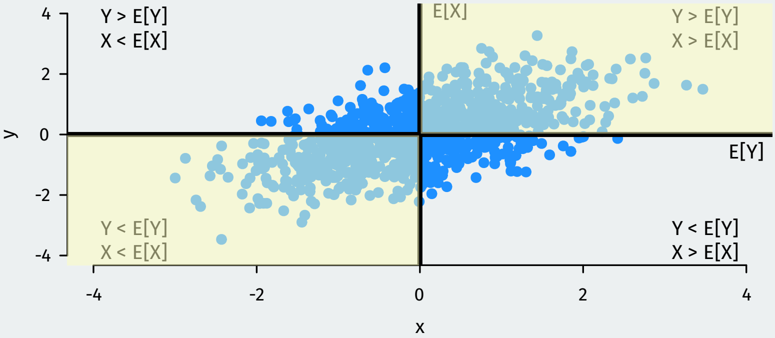

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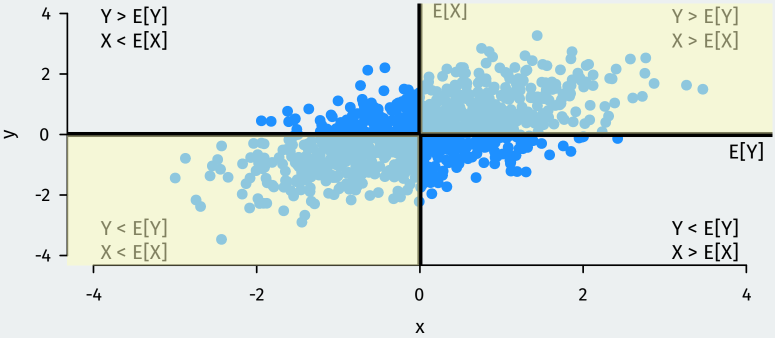

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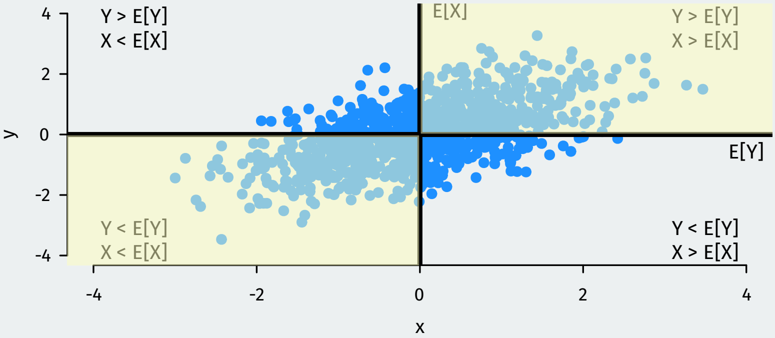

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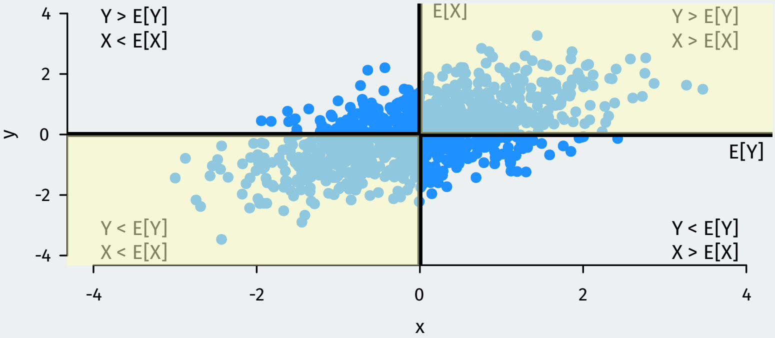

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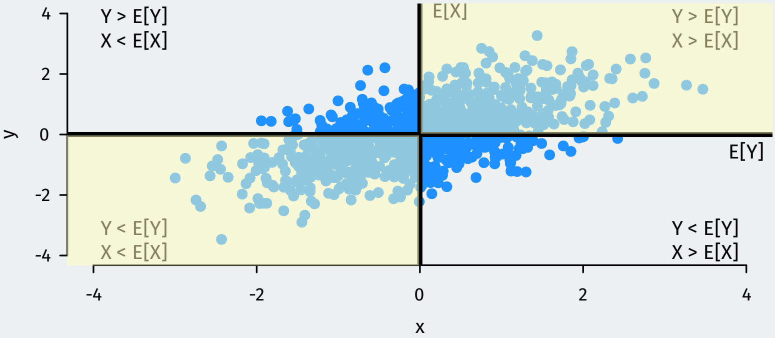

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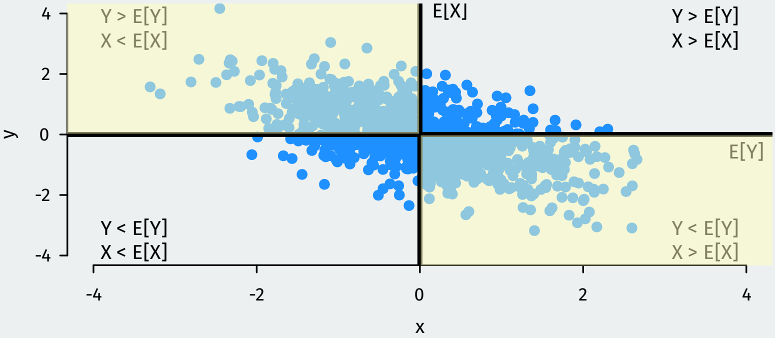

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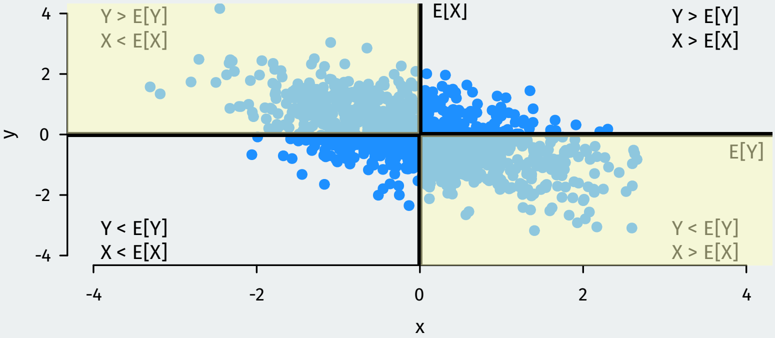

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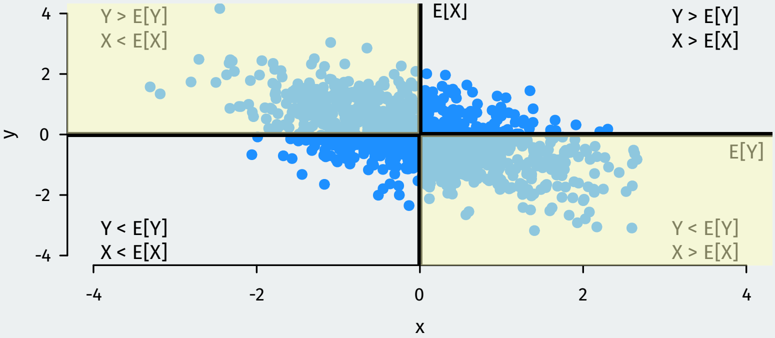

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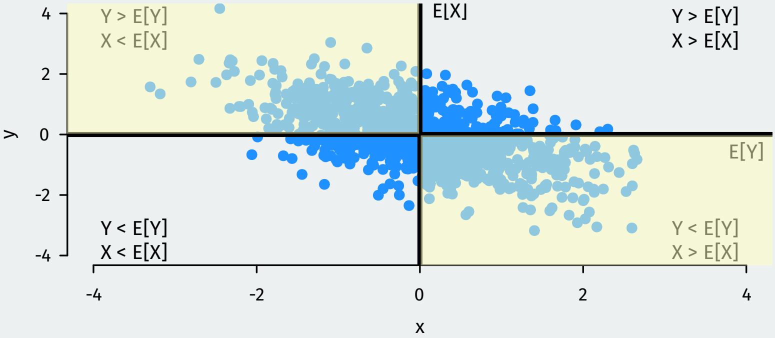

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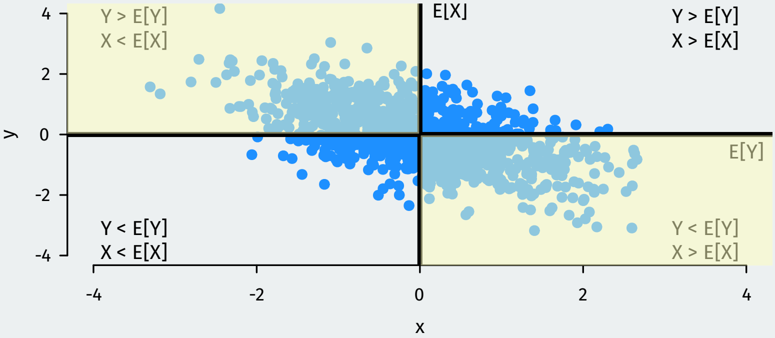

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$$\mathbb{V}[X_1 + \cdots + X_n] = \mathbb{V}[X_1] + \cdots + \mathbb{V}[X_n] + 2\sum_{i<j}\text{Cov}(X_i, X_j)$$

- If X and Y independent, $\mathbb{V}[X+Y] = \mathbb{V}[X] + \mathbb{V}[Y]$.
    - Beware: $\mathbb{V}[X-Y] = \mathbb{V}[X] + \mathbb{V}[Y]$ as well.

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- Covariance is a measure of **linear dependence**, so it can miss non-linear dependence.

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  - $|\rho(X, Y)| = 1$ if and only if X and Y are perfectly correlated with a deterministic linear relationship: $Y = a + bX$.

4/ Random vectors

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- Expectation of a random vector is just the vector of expectations:

$$\mathbb{E}[\mathbf{X}] = (\mathbb{E}[X_1], \mathbb{E}[X_2], \dots, \mathbb{E}[X_k])'$$

Covariance matrices

- Covariance matrix generalizes (co)variance to this setting:

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- We usually write $\mathbb{V}[\mathbf{X}] = \mathbf{\Sigma}$ and it is a $k \times k$ **symmetric** matrix:

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- Symmetric ($\mathbf{\Sigma} = \mathbf{\Sigma}'$) because $\text{Cov}(X_i, X_j) = \text{Cov}(X_j, X_i)$.

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    - $\mathbf{I}_k$ is the k by k identity matrix because $\mathbb{V}[Z_j] = 1$ and $\text{Cov}(Z_i, Z_j) = 0$.

Linear transformations of random vectors

Theorem

If $\mathbf{X} \in \mathbb{R}^k$ with $k \times 1$ expectation $\boldsymbol{\mu}$ and $k \times k$ covariance matrix $\boldsymbol{\Sigma}$, and $\mathbf{A}$ is a $q \times k$ matrix, then $\mathbf{AX}$ is a random vector with mean $\mathbf{A}\boldsymbol{\mu}$ and covariance matrix $\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}'$.

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    - $\boldsymbol{\mu}$: $q \times 1$ mean vector $\mathbb{E}[\mathbf{X}] = \boldsymbol{\mu}$
    - $\mathbb{V}[\mathbf{X}] = \mathbf{BB}'$: $q \times q$ covariance matrix.
- More generally, if $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ then $\mathbf{Y} = \mathbf{a} + \mathbf{BX} \sim \mathcal{N}(\mathbf{a} + \mathbf{B}\boldsymbol{\mu}, \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}')$

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- If (X, Y) are multivariate normal with $\text{Cov}(X, Y) = 0$, then X and Y are independent.