4: Expectation

Fall 2023

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Gov 2002 (Harvard)

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- Today: begin to summarize distributions with a few numbers.

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- How should we summarize the distribution of causal effects?

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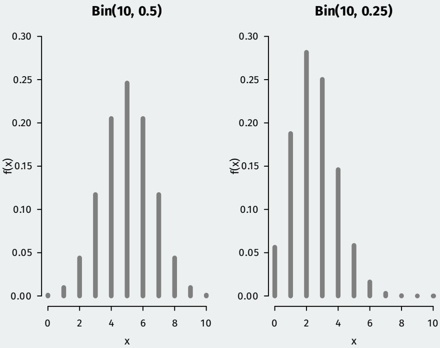

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    - but we'll use our sample to learn about them

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  - We'll use this intuition to create an average/mean for r.v.s.

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  - Converse isn't true!

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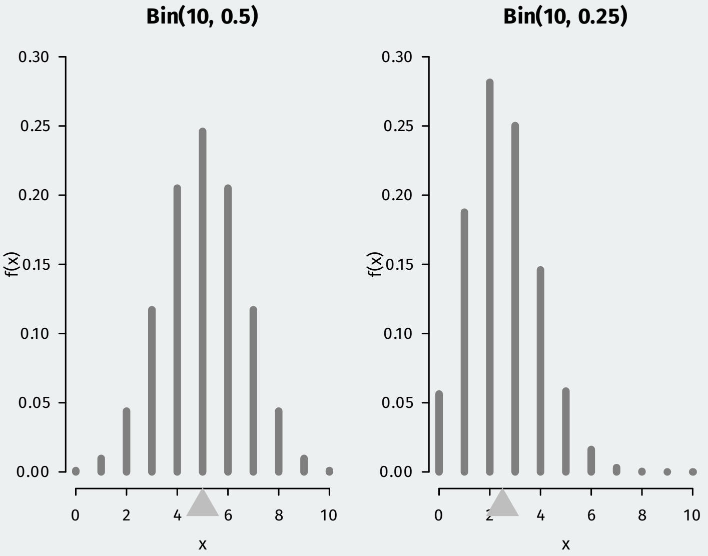
Expectation as balancing point



2/ Linearity of Expectations

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- Use linearity:

$$\mathbb{E}[X] = \mathbb{E}[X_1 + \cdots + X_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n] = np$$

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- Intuition: on average, the sample mean is equal to the population mean.

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  - If $X \geq Y$ with probability 1, then $\mathbb{E}(X) \geq \mathbb{E}(Y)$.

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  - Risk avoidance/concave utility $U = Y^{1/2} \rightsquigarrow \mathbb{E}[U(Y)] \approx 2.41$

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- Often, both of these are assumed away by assuming $\mathbb{E}[|X|] < \infty$ which implies $\mathbb{E}[X]$ exists and is finite.

3/ Indicator Variables

Indicator variables/fundamental bridge

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$$\mathbb{P}(A_1 \cup \cdots \cup A_n) \leq \mathbb{P}(A_1) + \cdots + \mathbb{P}(A_n)$$

  - Use the fact that $\mathbb{I}(A_1 \cup \cdots \cup A_n) \leq \mathbb{I}(A_1) + \cdots + \mathbb{I}(A_n)$ and then take expectations.

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$$\begin{aligned}
\mathbb{E}[I_j] &= \mathbb{P}(\text{cond } j \text{ empty}) \\
&= \mathbb{P}(\{\text{unit 1 not in cond } j\} \cap \cdots \cap \{\text{unit } n \text{ not in cond } j\}) \\
&= \mathbb{P}(\{\text{unit 1 not in cond } j\}) \cdots \mathbb{P}(\{\text{unit } n \text{ not in cond } j\}) \\
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- Thus, we have $\mathbb{E}\left[\sum_j I_j\right] = k(1 - 1/k)^n$.

4/ Variance

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- Useful equivalent representation of the variance:

$$\mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

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- Example: $\mathbb{E}[X^2]$ where $X \sim \text{Bin}(n, p)$.

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Example - number of treated units

- Use LOTUS to calculate the variance for a discrete r.v.:

$$\mathbb{V}[X] = \sum_{j=1}^{k} (x_j - \mathbb{E}[X])^2 \mathbb{P}(X = x_j)$$

x	$p_X(x)$
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    - But this doesn't hold for dependent r.v.s
4. $\mathbb{V}[X] \geq 0$ with equality holding only if X is a constant, $\mathbb{P}(X = b) = 1$.

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- Binomials are the sum of **independent** Bernoulli r.v.s so:

$$\mathbb{V}[X] = \mathbb{V}[X_1 + \cdots + X_n] = \mathbb{V}[X_1] + \cdots + \mathbb{V}[X_n] = np(1-p)$$

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  - We don't know what distribution it takes though!

5/ Inequalities

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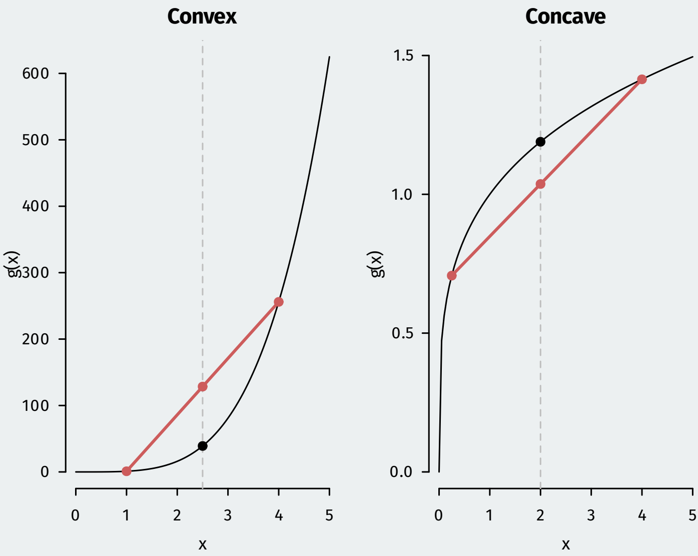
- Remember that $\mathbb{E}[a + bX] = a + b\mathbb{E}[X]$ is linear, but $\mathbb{E}[g(X)] \neq g(\mathbb{E}[X])$ for nonlinear functions.
- Can we relate those? Yes for **convex** and **concave** functions.

Concave and convex



Jensen's inequality

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Let X be a r.v. Then, we have

$$\mathbb{E}[g(X)] \geq g(\mathbb{E}[X]) \qquad \text{if } g \text{ is convex}$$
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with equality only holding if g is linear.

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    - $\mathbb{E}[1/X] \geq 1/\mathbb{E}[X]$
    - $\mathbb{E}[\log(X)] \leq \log(\mathbb{E}[X])$

6/ Poisson Distribution

Poisson

Definition

An r.v. X has the **Poisson distribution** with parameter $\lambda > 0$, written $X \sim \text{Pois}(\lambda)$ if the p.m.f. of X is:

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    - Number of contributions a candidate for office receives in a day.
- Key calculus fact that makes this a valid p.m.f.: $\sum_{k=0}^{\infty} \lambda^k / k! = e^{\lambda}$.

Poisson properties

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- If $X \sim \text{Bin}(n, p)$ with n large and p small, then X is approx $\text{Pois}(np)$.