

# 1: Basic Probability

Fall 2023

Matthew Blackwell

Gov 2002 (Harvard)



What is a reasonably safe gathering size in the age of COVID?

# Probability and assumptions

- If we have a gathering of size  $n$  drawn randomly from population of MA, what's the probability someone in attendance is infected?
- Calculating probabilities relies on **assumptions**
  - Imagine current incidence rate of COVID in MA: 1.37%.
  - Gathering is a random sample from population.
  - Wrong assumptions  $\rightsquigarrow$  wrong probabilities.
- Why do we care? Enables learning from data!
  - Are the data consistent with the probabilities?
  - If not, casts doubt on our assumptions  $\rightsquigarrow$  we've learned something!
  - The heart of **statistical inference**

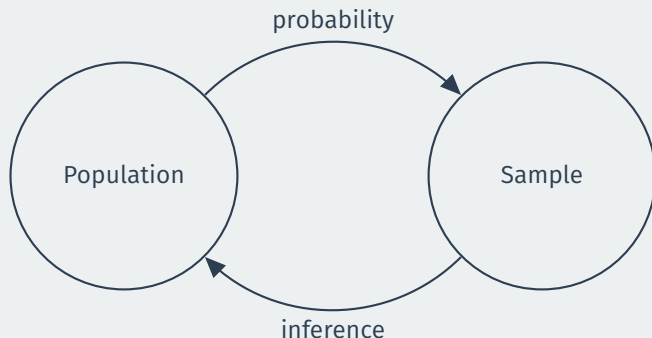
# Conjunction fallacy

- Why mathematical probability?
  - Our intuitions about probability are often **terrible**.

Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations.

- What is more probable?
  1. Linda is a bank teller?
  2. Linda is a bank teller and is active in the feminist movement?
- Famous example of the **conjunction fallacy** called the Linda problem.
  - Majority of respondents chose 2, but this is impossible!
- Learning mathematical probability avoids these mistakes!

# Learning about populations



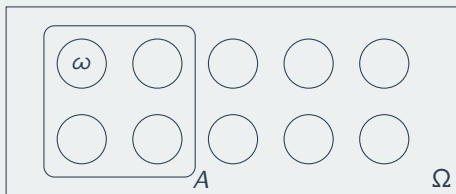
- **Probability:** formalize the uncertainty about how our data came to be.
- **Inference:** learning about the population from a set of data.

# Roadmap

1. Naive Definition of Probability
2. Non-naive Definition of Probability

# 1/ Naive Definition of Probability

# Sample spaces & events



- Probability formalizes chance variation or uncertainty in outcomes.
  - It might rain or be sunny today, we don't know which.
  - To formalize, we need to define the set of possible outcomes.
- A **sample space**  $\Omega$  is the set of possible outcomes.
  - Can be finite, countably infinite, or uncountably infinite.
- $\omega \in \Omega$  is one particular **outcome**.
- A subset of  $\Omega$  is an **event** and we write this as  $A \subset \Omega$ .



# Naive definition of probability

- Naive definition of probability: **all outcomes equally likely**, so:

$$\mathbb{P}(A) = \frac{\text{number of elements in } A}{\text{number of elements in } \Omega}$$

- Often wrong, but justified under a few DGPs:
  - **Symmetry**: fair coin, shuffled cards, dice, etc.
  - **Simple random sample** from a population.
- Example: randomly select a card from a standard deck of playing cards
  - Each of the 52 card has equal probability

$$\mathbb{P}(4\clubsuit) = \mathbb{P}(4\heartsuit) = 1/52$$

# Counting

- **Multiplication rule:** if you have two sub-experiments,  $A$  with  $a$  possible outcomes and  $B$  with  $b$  possible outcomes, then in the combined experiment there are  $ab$  possible outcomes.
- Example: what to watch where?
  - What to watch? Netflix or Hulu.
  - Where to watch? TV, tablet, or phone.
  - $2 \times 3 = 6$  possible outcomes (Netflix on TV, Hulu on phone, ...)
- Assumes number of outcomes in one experiment doesn't depend on the outcome of the other experiment.

# Sampling objects

- **Sample with replacement:** Choose  $k$  objects from a set of  $n$  one at a time with replacement.
  - Any object may be selected multiple times.
  - There are  $n^k$  possible outcomes when order matters (multiplication rule)
- **Sampling without replacement:** Choose  $k$  objects from a set of  $n$  one at a time without replacement.
  - Chosen object can't be chosen again.
  - Number of possibilities:  $n(n-1)\cdots(n-k+1)$
- Ex: 2020 IA caucuses had 11 candidates. How many top-3 possibilities?
  - 11 first-place choices
  - 10 second-place choices among the remaining candidates
  - 9 third-place choices
  - Total:  $11 \cdot 10 \cdot 9 = 990$  possibilities

# Avoiding overcounting

- What if order doesn't matter? How many subsets of size  $k$  are there?
- If each possibility occurs  $c$  times, we can divide by  $c$ .
- What if we wanted to book 3 IA caucus candidates for a debate?
  - Multiplication rule would count both (Biden, Warren, Sanders) and (Sanders, Biden, Warren).
  - But both of those make the same debate stage.
  - By the multiplication rule, there are  $3 \cdot 2 \cdot 1 = 6$  ways to arrange them.
- **Binomial coefficient:** number of subsets of size  $k$  in a group of  $n$  objects:

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!}$$

# The lady tasting tea

## Lady tasting tea

Your friend asks you to grab a tea with milk for her before meeting up and she says that she prefers tea poured before the milk. You stop by a local tea shop and ask for a tea with milk. When you bring it to her, she complains that it was prepared milk-first.

- You're skeptical that she can tell the difference, so you devise a test:
  - Prepare 8 cups of tea, 4 milk-first, 4 tea-first
  - Present cups to friend in a **random** order
  - Ask friend to pick which 4 of the 8 were milk-first.

# Assuming we know the truth

- Friend picks out all 4 milk-first cups correctly!
- Statistical thought experiment: how often would she get all 4 correct **if she were guessing randomly?**
  - Only one way to choose all 4 correct cups.
  - But  $\binom{8}{4} = 70$  ways of choosing 4 cups among 8.
  - Choosing at random  $\approx$  picking each of these 70 with equal probability.
- Chances of guessing all 4 correct is  $\frac{1}{70} \approx 0.014$  or 1.4%.
- $\rightsquigarrow$  the guessing hypothesis might be implausible.

# Birthday problem

## Birthday problem

There are  $k$  people in a room. Assume each person's birthday is equally likely to be any of the 365 days of the year (no leap babies) and birthdays are independent. What is the probability that at least one pair of people have the same birthday?

## **2/** Non-naive Definition of Probability



- A **probability space** consists of:
  - Sample space  $\Omega$
  - Probability function  $\mathbb{P}$  mapping events  $A \subseteq \Omega$  onto the real line.
- The function  $\mathbb{P}$  must satisfy the following axioms:
  1. (Non-negativity)  $\mathbb{P}(A) \geq 0$  for every event  $A$
  2. (Normalization)  $\mathbb{P}(\Omega) = 1$
  3. (Additivity) If a series of events,  $A_1, A_2, \dots$ , are disjoint, then

$$\mathbb{P}\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mathbb{P}(A_j).$$

- Probability function assigns “mass” to regions of the sample space.

# Interpretation of probabilities

- How do we interpret  $\mathbb{P}(A)$ ? Huge debate about this in stats literature.
1. **Frequentist:**  $\mathbb{P}()$  reflect relative frequency in a large number of trials.
    - Repeat a coin flip many times  $\rightsquigarrow$  frequency of head  $\approx 0.5$ .
  2. **Bayesian**  $\mathbb{P}()$  are subjective beliefs about outcomes.
    - How likely I think a particular event will be.
- Set drama aside: both viewpoints are helpful in different contexts.
    - Properties of probabilities exactly the same in either approach.
    - This class: focus on frequentist perspectives because it's pervasive.

# Gambling 102

- What's the probability of selecting a 4 card from a well-shuffled deck?
  - “Well-shuffled”  $\rightsquigarrow$  “randomly selected”  $\rightsquigarrow$  all cards have prob.  $1/52$
- “4 card” event =  $\{4\clubsuit \cup 4\spadesuit \cup 4\heartsuit \cup 4\diamondsuit\}$
- Union of mutually exclusive events  $\rightsquigarrow$  use additivity
  - $\rightsquigarrow \mathbb{P}(4 \text{ card}) = \mathbb{P}(4\clubsuit) + \mathbb{P}(4\spadesuit) + \mathbb{P}(4\heartsuit) + \mathbb{P}(4\diamondsuit) = \frac{4}{52}$

# Some properties of probabilities

1.  $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$

- Probability of not  $A$  is 1 minus the probability of  $A$ .
- Follows from  $A \cup A^c = \Omega$  and  $1 = \mathbb{P}(\Omega) = \mathbb{P}(A) + \mathbb{P}(A^c)$

2. If  $A \subset B$ , then  $\mathbb{P}(A) \leq \mathbb{P}(B)$






- Subsets of events have lower probability than the event.
- Probability of 5 SCOTUS votes in the liberal direction is less than the probability of liberals winning the case ( $A = \{5, 6, 7, 8, 9\}$ )

















































3.  $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$





- Avoid “double-counting” the part where  $A$  and  $B$  overlap.
- **Inclusion-exclusion**

# Appendix

# Gambling

- A standard deck of playing cards has 52 cards:
  - 13 rank cards: (2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A)
  - in each of 4 suits: (, , , )
- Hypothetical experiment: pick a card, any card.
- One possible outcome: picking a 4
- Sample space:

2 3 4 5 6 7 8 9 10 J Q K A  
2 3 4 5 6 7 8 9 10 J Q K A  
2 3 4 5 6 7 8 9 10 J Q K A  
2 3 4 5 6 7 8 9 10 J Q K A

- An event: picking a Queen,  $\{Q, Q, Q, Q\}$

# Social science examples

- Examples of substantively interesting **sample spaces**:
  - House elections: incumbents win or lose  $\Omega = \{W, L\}$ .
  - Supreme Court votes in a liberal direction:  $\Omega = \{0, 1, \dots, 9\}$ .
  - Voter turnout: percent turnout in some county:  $\Omega = [0, 100]$ .
  - Duration of a war: any nonnegative number,  $\Omega = [0, \infty)$ .
- **Events** from these examples:
  - House election: incumbent wins,  $A = W$
  - Supreme Court: liberals win a Supreme Court case,  $A = \{5, 6, 7, 8, 9\}$
  - Voter turnout: over half the population votes,  $A \in (50, 100]$
  - Duration of war: war ends within five years,  $A \in [0, 5)$
- We want to know or model the probability of these events!

# New events from old events

- Define events/sets  $A, B \subseteq \Omega$ .
- **Complement:**  $A^c$  (“not  $A$ ”)
  - All of the outcomes in  $\Omega$  not in  $A$ .
  - Complement of picking a **red card** is picking a **black card**.
  - $\Omega^c = \emptyset$ , where  $\emptyset$  is the empty set—nothing happens.
- **Union** of two events  $A \cup B$  (“ $A$  or  $B$ ”)
  - The event that  $A$  or  $B$  occurs.
  - $\text{Queen} \cup \clubsuit = \text{any club card or any queen card}$ .
- **Intersection**  $A \cap B$  or just  $AB$  (“ $A$  and  $B$ ”)
  - The event that both  $A$  and  $B$  occur
  - $\text{Queen} \cap \clubsuit = Q\clubsuit$  (queen of clubs)
- **De Morgan’s laws:**  $(A \cup B)^c = A^c \cap B^c$  and  $(A \cap B)^c = A^c \cup B^c$ .
  - not  $Q \cup \clubsuit$  must be not  $Q$  and not  $\clubsuit$
  - not  $Q \cap \clubsuit = Q\clubsuit$  is either not a  $Q$  or not a  $\clubsuit$



# Relationships between events

- $A$  **implies**  $B$  when  $A \subseteq B$ .
  - $A = 4\clubsuit$  and  $B = \clubsuit$  card
- **Disjoint/mutually exclusive:**  $A \cap B = \emptyset$ .
  - $A = \clubsuit$  card,  $B = \spadesuit$  card
- A **partition** is a set of mutually disjoint events whose union is  $\Omega$ .
  - $A_1 = \clubsuit, A_2 = \spadesuit, A_3 = \diamond, A_4 = \heartsuit$
  - $A_1, A_2, A_3, A_4$  is a partition of a 52-card deck